International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

Sequential Estimation of the Mean Survival Time of the Exponential Distribution

Sandeep Bhougal
School of Mathematics, Shri Mata Vaishno Devi University, Kakryal, Jammu (J & K).

Abstract

The problem of fixed-width Confidence Interval for the mean survival time is considered. Sequential procedures are adopted based on the maximum likelihood estimators (MLE) and uniform minimum variance unbiased estimators (UMVUE) of the scale parameter. A comparative study of the two sequential procedures is done and second-order approximations are obtained and they are proved to be 'asymptotically efficient and consistent.'

***Keywords*--Exponential Distribution, Sequential Estimation,**

I. INTRODUCTION

Exponential distribution plays an important part in life-testing and reliability problems and it is the simplest and the most widely exploited model in this area. Early work by Sukhatme (1973) and later work by Epstein and Sobel (1953, 1954, 1955) and Epstein (1954, 1960) gave numerous results and popularized the exponential as a lifetime distribution, especially in the area of industrial life testing. Sequential techniques have been utilized by several researchers to deal with various inferential problems related to one-parameter and two-parameter exponential distributions. For some citations one may refer to Basu (1971), Starr and Woodroofe (1972), Mukhopadhay (1974), Mukhopadhay and Hilton (1986), Chaturvedi and Shukla (1990), Chaturvedi (1996), Manisha, P., M.M. Ali and J. woo (2005) and Gupta and Bhougal (2006).

In this paper, we consider the problem of constructing fixed-width confidence interval for the mean survival time, for addressing which in section 3, we consider the problem of sequential interval estimation. Sequential procedures are adopted based on the maximum likelihood estimator (MLE) and uniformly minimum variance unbiased estimator (UMVUE) of the scale parameter. A comparative study of the two sequential procedures is done and they are proved to be 'asymptotically efficient and consistent.' In section 4, the problem of sequential point estimation of the mean survival time is tackled. Consideration is given to squared-error loss function and linear cost of sampling.

Two sequential procedures (one based on the MLE and the other based on the UMVUE of the scale parameter) are proposed. Second-order approximations are obtained and a comparative study is done.

II. THE SET-UP OF THE ESTIMATION PROBLEMS

Let $\{X_i\}_{i=1,2,.....}$ be a sequence of independent and identically distributed (i.i.d.) random variables from two-parameter exponential distribution having the probability density function p.d.f. given by

$$f(x;\mu,\sigma)=\frac{1}{\sigma}\exp\left(-\frac{(x-\mu)}{\sigma}\right);\ x>\mu,\ \sigma>0 \quad (2.1)$$

Both $\mu\in(-\infty,\infty)$ [1] and $\sigma\in(0,\infty)$ are unknown. Have been recorded a random sample $X_1,.....,X_n$ of size $n(\geq 2)$, the MLE's of μ and σ are $X_{n(1)}=\min(X_1,.....,X_n)$ and $\hat{\sigma}_n=n^{-1}\sum_{i=1}^{n}(X_i-X_{n(1)})$, respectively and the UMVUE of σ is $\hat{\sigma}_n^*=(n-1)^{-1}\sum_{i=1}^{n}(X_i-X_{n(1)})$.

Our first estimation problem is to construct fixed-width confidence interval for the mean survival time. For the model (2.1), the mean survival time is $E(X)=\mu+\sigma=\theta$.

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

For pre-assigned $d>0$ and $\alpha \in (0,1)$, suppose one wishes to construct a Confidence Interval for θ having width $2d$ and coverage probability atleast $1-\alpha$. The MLE, as well as, the UMVUE of θ is $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$.

We define $I_n = \left[\overline{X}_n - d\,\overline{X}_n + d,\right]$ Using the facts that $E\left(\overline{X}_n\right)=\theta,\ Var\left(\overline{X}_n\right)=\frac{\sigma^2}{n}$ and applying the central limit theorem (CLT), we conclude that

$$\frac{\left(\overline{X}_n-\theta\right)}{\left(\sigma/\sqrt{n}\right)} \xrightarrow{L} N(0,1). \qquad (2.2)$$

Using (2.2) and denoting by $\Phi(y)$, the cumulative distribution function (c.d.f.) of the standard normal variate (SNV), we get

$$P[\theta \in I_n] = P\left[\frac{\left|\overline{X}_n-\theta\right|}{\left(\sigma/\sqrt{n}\right)} \le \frac{d\sqrt{n}}{\sigma}\right]$$

$$\doteq 2\Phi\left(\frac{d\sqrt{n}}{\sigma}\right)-1. \qquad (2.3)$$

Let ‘a’ be the constant defined by

$$2\Phi(a)-1=1-\alpha. \qquad (2.4)$$

Using the monotonicity property of the c.d.f., it follows from (2.3) and (2.4) that, in order to achieve $P[\theta \in I_n] \ge 1-\alpha$, the sample size required is the smallest positive integer $n \ge n_0$.

Where

$$n_0 = \left(\frac{a}{d}\right)^2 \sigma^2. \qquad (2.5)$$

However, in the absence of any knowledge about σ, no fixed sample size procedure achieves the goals of ‘preassigned width and coverage probability’ simultaneously for all values of σ. In such a situation, motivated by (2.5), in Section 3, we develop sequential procedures based on the MLE and UMVUE of σ.

Our second estimation problem is the minimum risk point estimation of θ. Let the loss incurred in estimating θ by $\overline{X}_n$ be squared-error plus linear cost of sampling, that is,

$$L\left(\theta,\overline{X}_n\right) = A\left(\overline{X}_n-\theta\right)^2 + n, \qquad (2.6)$$

Where $A(>0)$ is the known weight. The risk corresponding to the loss function (2.6) is

$$R_n(A) = \frac{A\sigma^2}{n} + \theta. \qquad (2.7)$$

Treating n as a continuous variable, the value n^* of n minimizing the risk (2.7) is

$$n^* = A^{1/2}\sigma, \qquad (2.8)$$

and substituting $n=n^*$ in (2.7), the corresponding minimum risk is

$$R_{n^*}(A) = 2n^*. \qquad (2.9)$$

But, in the absence of any knowledge about σ, no fixed sample size procedure minimizes the risk for all values of σ. As a solution to the problem, in conformity with (2.8), in Section 4, we propose sequential procedure based on the MLE and UMVUE of σ.

III. Sequential Procedures For Fixed-Width Confidence Interval Estimation Of The Mean Survival Time

We first consider the sequential procedure based on the MLE of σ.

Let us start with a sample of size $m(\geq 2)$. Then, the stopping time $N_1 \equiv N_1(d)$ is the smallest positive integer $n_1 \geq m$ such that

$$n_1 \geq \left(\frac{a}{d}\right)^2 \hat{\sigma}_{n_1}^2$$

After stopping with N_1 observations, we construct the interval

$$I_{N_1} = \left[\overline{X}_{N_1} - d, \overline{X}_{N_1} + d\right] \text{ for } \theta.$$

In the following theorem, we prove that the sequential procedure (3.1) is 'asymptotically efficient and consistent' in Chow-Robbins (1965) sense.

Theorem 1

N_1 terminates with probability one, (3.2)

$$\lim_{d\to 0} N_1 = \infty, \quad (3.3)$$

$$\lim_{d\to 0} \left(\frac{N_1}{n_0}\right) = 1 \text{ a.s.}, \quad (3.4)$$

$$\lim_{d\to 0} E\left(\frac{N_1}{n_0}\right) = 1, \quad \text{'asymptotic efficiency'} \quad (3.5)$$

and

$$\lim_{d\to 0} P\left(\theta \in I_{N_1}\right) = 1 - \alpha, \text{ 'asymptotic consistency'} \quad (3.6)$$

Proof: Using the fact that $\frac{2n_1\hat{\sigma}_{n_1}}{\sigma} \overset{d}{=} \chi^2_{2(n_1-1)}$, it follows from (3.1) that

$$P(N_1 > n) \leq P\left[n_1 \leq \left(\frac{a}{d}\right)^2 \hat{\sigma}_{n_1}^2\right] = P\left[\hat{\sigma}_{n_1} > \left(\frac{d}{a}\right) n_1^{1/2}\right]$$

$$= P\left[\chi^2_{2(n_1-1)} > 2n_1\left(\frac{n_1}{n_0}\right)^{1/2}\right]$$

$$= P\left[Z_{n_1} > \frac{2n_1\left(\frac{n_1}{n_0}\right)^{1/2} - 2(n_1-1)}{\sqrt{4(n_1-1)}} \quad (3.1)\right] \quad (3.7)$$

Where $Z_{n_1} = \frac{\left\{\chi^2_{2(n_1-1)} - 2(n_1-1)\right\}}{\sqrt{4(n_1-1)}}$.

Since $Z_{n_1} \xrightarrow{L} Z$ as $n_1 \to \infty$, where Z is a Standard Normal Variate (SNV) and from Zacks (1971, p.561), $1 - \Phi(x) \approx x^{-1}\varphi(x)$ as $x \to \infty$, where $\varphi(.)$ stands for the p.d.f. of a SNV, we obtain from (3.7) that

$$P(N_1 > n_1) \leq 1 - \Phi\left(\frac{2n_1\left(\frac{n_1}{n_0}\right)^{1/2} - 2(n_1-1)}{2\sqrt{(n_1-1)}}\right)$$

Or

$$P(N_1 > n_1) \to \left[\left\{\frac{2n_1\left(\frac{n_1}{n_0}\right)^{1/2} - 2(n_1-1)}{2\sqrt{(n_n-1)}} \quad (3.6)\right\}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left[2n_1\left(\frac{n_1}{n_0}\right)^{1/2} - 2(n_1-1)\right]}$$

$\to 0$ *as* $n_1 \to \infty$, hence the result (3.2) follows.

Result (3.3) follows from the definition of N_1 given at (3.1).

From (3.1), we notice the inequality

$$\left(\frac{a}{d}\right)^2 \hat{\sigma}_{N_1}^2 \leq N_1 \leq \left(\frac{a}{d}\right)^2 \hat{\sigma}_{N_1}^2 + (m-1)$$

or

$$\frac{\left(\frac{a}{d}\right)^2 \hat{\sigma}_{N_1}^2}{\left(\frac{a}{d}\right)^2 \sigma^2} \leq \frac{N_1}{n_0} \leq \frac{\left(\frac{a}{d}\right)^2 \hat{\sigma}_{N_1}^2}{\left(\frac{a}{d}\right)^2 \sigma^2} + \frac{(m-1)d^2}{a^2\sigma^2}$$

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

or $$\frac{\hat{\sigma}^2_{N_1}}{\sigma^2} \le \frac{N_1}{n_0} \le \frac{\hat{\sigma}^2_{N_1}}{\sigma^2} + \frac{(m-1)}{n_0}. \qquad (3.8)$$

Utilizing (3.3), the fact that $\underset{N_1\to\infty}{Lim}\,\hat{\sigma}_{N_1} = \sigma \quad a.s.$. (Since $\hat{\sigma}_n$ is a consistent estimator of σ) and taking the limit of (3.8) throughout as $d \to 0$, we get

$$1 \le \underset{d\to 0}{Lim}\inf \frac{N_1}{n_0} \le \underset{d\to 0}{Lim}\sup \frac{N_1}{n_0} \le 1,$$

Hence the result (3.4) follows.

Let, for $0 < \in < 1$, $\theta_1 = (1-\in)n_0$ and $\theta_2 = (1+\in)n_0$. Applying Markov's inequality, we get

$$P(N_1 > \theta_1) \le \frac{E(N_1)}{\theta_1}$$

or $$E\left(\frac{N_1}{n_0}\right) \ge (1-\in)P(N_1 > \theta_1)$$

or $$E\left(\frac{N_1}{n_0}\right) \ge (1-\in)P\left(\frac{N_1}{n_0} > (1-\in)\right). \qquad (3.9)$$

Since $\in$ is arbitrary, application of (3.4) to (3.9) leads us to

$$\underset{d\to 0}{Lim}\inf E\left(\frac{N_1}{n_0}\right) \ge 1. \qquad (3.10)$$

Furthermore, we can write

$$E(N_1) = \sum_{n_1=m}^{\infty} n_1 P(N_1 = n_1)$$
$$\le \theta_2 P(m \le N_1 \le \theta_2) + \sum_{n_1 \ge \theta_2} (n_1+1)P(N_1 = n_1+1),$$

or, $$E\left(\frac{N_1}{n_0}\right) \le (1+\in) + \frac{1}{n_0}\sum_{n_1\ge\theta_2}(n_1+1)P(N_1 = n_1+1) \qquad (3.11)$$

Let us denote by

$$T(\theta_2) = \sum_{n_1\ge\theta_2}(n_1+1)P(N_1 = n_1+1).$$

It follows from the definition of N_1 given at (3.1) that

$$T(\theta_2) \le \sum_{n_1\ge\theta_2}(n_1+1)P\left[n_1 < \left(\frac{a}{d}\right)^2 \hat{\sigma}^2_{n_1}\right]$$

$$= \sum_{n_1\ge\theta_2}(n_1+1)P\left[\chi^2_{2(n_1-1)} > 2n_1\left(\frac{n_1}{n_0}\right)^{\frac{1}{2}}\right],$$

Which on applying exponential bounds leads us to

$$T(\theta_2) \le$$
$$\sum_{n_1\ge\theta_2}(n_1+1)\inf_{0<h<1/2}\left[\exp\left\{-2hn_1\left(\frac{n_1}{n_0}\right)^{\frac{1}{2}}\right\}E\left\{\exp\left(h\chi^2_{2(n_1-1)}\right)\right\}\right].$$

$$\le \sum_{n_1\ge\theta_2}(n_1+1)\inf_{0<h<1/2}\left[\exp\left\{-2hn_1\left(\frac{\theta_2}{n_0}\right)^{\frac{1}{2}}\right\}(1-2h)^{-(n_1-1)}\right]$$

$$\le \sum_{n_1\ge\theta_2}(n_1+1)\inf_{0<h<1/2}\left[\exp\left\{-2hn_1(1+\in)^{\frac{1}{2}}\right\}(1-2h)^{-n_1}\right]. \qquad (3.12)$$

This inequality is also valid for the value h_0 of h, which minimizes the function $E(N_1) = \sum_{n_1=m}^{\infty} n_1 P(N_1 = n_1)$

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

$$g(h) = \exp\left[-2hn_1(1+\in)^{\frac{1}{2}}\right](1-2h)^{-n_1}, \text{ i.e.}$$

$h_0 = \left(\frac{1}{2}\right)\left[1-(1+\in)^{-\frac{1}{2}}\right]$ and substituting this value of h_0 in (3.12), we get

$$T(\theta_2) \le \sum_{n_1 \ge \theta_2} (n_1+1)\left[(1+\in)^{\frac{1}{2}} \exp\left\{1-(1+\in)^{\frac{1}{2}}\right\}\right]^{n_1}$$

$$= \sum_{n_1 \ge \theta_2} b_{n_1}, \quad \text{say.} \qquad (3.13)$$

Since $\underset{n_1\to\infty}{Lim}\, b_{n_1}^{\frac{1}{n_1}} = (1+\in)^{\frac{1}{2}} \exp\left\{1-(1+\in)^{\frac{1}{2}}\right\} < 1$, the series involved on the right hand side of (3.13) is convergent. Hence we conclude that, for a positive constant k,

$$T(\theta_2) \le k. \qquad (3.14)$$

Utilizing (3.14), it follows from (3.11) that

$$\underset{d\to 0}{Lim\,Sup}\; E\left(\frac{N_1}{n_0}\right) \le 1. \qquad (3.15)$$

Result (3.5) now follows on combining (3.10) and (3.15).

Finally, we have

$$P(\theta \in I_{N_1}) = P\left[\frac{\left|\bar{X}_{N_1} - (\mu+\sigma)\right|}{\left(\frac{\sigma}{\sqrt{n_0}}\right)} \le \frac{d\sqrt{n_0}}{\sigma}\right]. \qquad (3.16)$$

We have shown [see (2.1)] that

$$\frac{\left\{\bar{X}_{n_0} - (\mu+\sigma)\right\}}{\left(\frac{\sigma}{\sqrt{n_0}}\right)} \xrightarrow{L} N(0,1) \text{ as } n_0 \to \infty. \qquad (3.17)$$

Application of (3.2), (3.3), (3.4) and Theorem 1 of Anscombe (1952) to (3.17) gives that

$$\frac{\left\{\bar{X}_{N_1} - (\mu+\sigma)\right\}}{\left(\frac{\sigma}{\sqrt{n_0}}\right)} \xrightarrow{L} N(0,1) \quad \text{as} \quad d \to 0. \qquad (3.18)$$

Since probability measure is bounded by unity, from (3.16), (3.18) and dominated convergence theorem, we get, for Z to be a Standard normal variate

$$\underset{d\to 0}{Lim}\, P(\theta \in I_{N_1}) = P\left[|Z| \le a\right] = 2\Phi(a) - 1 = 1-\alpha.$$

and (3.6) follows.

In the following theorem, we obtain the second-order approximations for the average sample number (ASN) corresponding to the sequential procedure (3.1).

Theorem 2

For all $m \ge 4$, as $d \to 0$,

$$E(N_1) = n_0 + 2v - 3.5 + o(1),$$

where v is specified.

Proof: Utilizing the fact that $\frac{2n_1\hat{\sigma}_{n_1}}{\sigma} = \sum_{j=1}^{n_1-1} Y_j$, with $Y_j \overset{d}{=} \chi^2_{(2)}$, we can re-write the stopping rule (3.1) as

$$N_1 = \inf\left[n_1 \ge m : \sum_{j=1}^{n_{1-1}} \left(\frac{Y_j}{2}\right) \le n_1^{3/2} n_0^{-1/2}\right].$$

Let us define a new stopping rule N_1^* as

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

$$N_1^* = \inf\left[n_1 \ge m-1 : \sum_{j=1}^{n_1}\left(\frac{Y_j}{2}\right) \le (n_1+1)^{3/2} n_0^{-1/2}\right].$$

$$= \inf\left[n_1 \ge m-1 : \sum_{j=1}^{n_1}\left(\frac{Y_j}{2}\right) \le n_1^{3/2}\left(1+n_1^{-1}\right)^{3/2} n_0^{-1/2}\right]. \quad (3.20)$$

It follows from a result of Swanepoel and Vanwyk (1982) that N_1 and N_1^* are identically distributed. Comparing (3.20) with equation (1.1) of Woodroofe (1977), we obtain in his notations,

$$\alpha = \frac{3}{2},\ \beta = 2\ \mu = E\left(\frac{Y_j}{2}\right) = 1,\ \tau^2 = \text{var}\left(\frac{Y_j}{2}\right) = 1,\ L_0 = \frac{3}{2},\ C = n_0^{-1/2}\ \textit{and}\ \lambda = n_0.$$

Moreover, denoting by $F(x)$, the c.d.f. of Y_j, we have

$$F(x) = P(Y_j \le x) = K\int_0^x e^{-y/2}\,dy \le Kx,$$

So that $a = 1$. Thus we obtain from Theorem 2.4 of Woodroofe (1977) that, for all $m \ge 4$, as $d \to 0$,

$$E\left(N_1^*\right) = n_0 + 2v - 4.5 + o(1).$$

Since $N_1^* = N_1 - 1$, we have

$$E(N_1) = n_0 + 2v - 3.5 + o(1).$$

and the theorem follows.

The following theorem provides the asymptotic distribution of the stopping time.

Theorem 3: As $d \to 0$,

$$(n_0)^{-\frac{1}{2}}(N_1 - n_0) \xrightarrow{L} N(0,4).$$

Proof: From the inequality (3.8),

$$\left(\hat{\sigma}^2_{N_1} - \sigma^2\right) \le \left(\frac{d}{a}\right)^2 (N_1 - n_0) \le \left(\hat{\sigma}^2_{N_1} - \sigma^2\right) + (m-1)\left(\frac{d}{a}\right)^2,$$

or

$$\frac{\sqrt{n_0}}{2\sigma^2}\left(\hat{\sigma}^2_{N_1} - \sigma^2\right) \le \frac{(N_1 - n_0)}{2\sqrt{n_0}} \le \frac{\sqrt{n_0}}{2\sigma^2}\left(\hat{\sigma}^2_{N_1} - \sigma^2\right) + (m-1)\left(\frac{d}{a}\right)^2 \quad (3.21)$$

We have,

$$E\left(\hat{\sigma}^2_{n_0}\right) = \frac{\sigma^2}{4n_o^2} E\left[\chi^2_{2(n_0-1)}\right]^2 = \sigma^2\left(1 - \frac{1}{n_0}\right) \quad (3.22)$$

and

$$E\left(\hat{\sigma}^4_{n_0}\right) = \frac{\sigma^4}{16n_o^4} E\left[\chi^2_{2(n_0-1)}\right]^4 = \sigma^4\left(1 + \frac{2}{n_0} - \frac{1}{n_0^2} - \frac{2}{n_0^3}\right) \quad (3.23)$$

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

From (3.22) and (3.23),

$$Var\left(\hat{\sigma}^2_{n_0}\right) = \frac{4\sigma^4}{n_0} + o\left(n_0^{-1}\right) \quad (3.24)$$

From (3.22), (3.24) and the CLT,

$$\frac{\sqrt{n_0}}{2\sigma^2}\left(\hat{\sigma}^2_{n_0} - \sigma^2\right) \xrightarrow{L} N(0,1), \text{ as } n_0 \to \infty. \quad (3.25)$$

It follows from (3.2), (3.3), (3.4), (3.25) and Theorem 1 of Anscombe (1952) that, as $d \to 0$,

$$\frac{\sqrt{n_0}}{2\sigma^2}\left(\hat{\sigma}^2_{N_1} - \sigma^2\right) \xrightarrow{L} N(0,1). \quad (3.26)$$

Application of (3.26) to (3.21) leads to the desired result.

Remarks 1:

One can use the technique of Bhattacharya and Mallik (1973) or Woodroofe (1977) in order to obtain the asymptotic distribution of stopping time. However, our method of obtaining the same is simple and direct. We can also obtain the result (3.5) from Theorem 4.2. But, it requires $m \geq 4$, whereas, (3.5) holds for all $m \geq 2$.

Now we consider the sequential procedure based on the UMVUE of σ.

We take $m(\geq 2)$ as the initial sample size. Then, the stopping time $N_2 \equiv N_2(d)$ is the smallest positive integer $n_2 \geq m$ such that

$$n_2 \geq \left(\frac{a}{d}\right)^2 \hat{\sigma}^{*2}_{n_2}. \quad (3.27)$$

After stopping with N_2 observations, we construct the confidence interval

$I_{N_2} = \left[\bar{X}_{N_2} - d, \bar{X}_{N_2} + d\right]$ for θ.

Now we state the following theorems, concerning various results for the stopping time N_2.

Theorem 4:

N_2 terminates with probability one

$$\lim_{d\to 0} N_2 = \infty,$$

$$\lim_{d\to 0}\left(\frac{N_2}{n_0}\right) = 1 \text{ a.s.},$$

$$\lim_{d\to 0} E\left(\frac{N_2}{n_0}\right) = 1, \text{ 'asymptotic efficiency'}$$

And

$$\lim_{d\to 0} P\left(\theta \in I_{N_2}\right) = 1-\alpha, \text{ 'asymptotic consistency'}$$

Proof: The proof is similar to that of Theorem 1.

Theorem 5: For all $m \geq 4$, as $d \to 0$,

$$E(N_2) = n_0 + 2v - 3 + 0(1),$$

where v is same as in Theorem 2.

Proof: The proof can be obtained along the lines of that of Theorem 2.

Remarks 2:

It is to be noted here that N_2 enjoys all the 'optimal' properties of N_1. However, if we compare Theorems 2 and 5, we conclude that ASN for N_2 is slightly higher than that of N_1.

Theorem 6: As $d \to 0$, $(n_0)^{-1/2}(N_2 - n_0) \xrightarrow{L} N(0,4)$.

Proof: The result can be obtained along the lines of that of Theorem 3.

IV. Sequential Procedures For The Point Estimation Of The Mean Survival Time

First of all, we consider sequential procedure based on the MLE of σ.

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

We begin with a sample of size $m(\geq 2)$. Then, the stopping time $N_1 \equiv N_1(A)$ is defined by

$$N_1 = \inf\left[n_1 \geq m : n_1 \geq A^{1/2}\hat{\sigma}_{n_1}\right]. \quad (4.1)$$

After stopping, we estimate θ by $\overline{X}_{N_1}$. The risk associated with the sequential procedure (4.1) is

$$R_{N_1}(A) = AE\left[\left(\overline{X}_{N_1} - \theta\right)^2\right] + E(N_1). \quad (4.2)$$

In what follows, we obtain second-order approximations for the risk corresponding to the sequential procedure (4.1). Before proving the main result, we establish some lemmas.

Lemma 1: For all $m \geq 3$, *as* $A \to \infty$,

$$E(N_1) = n^* - 1.253 + o(1).$$

Proof: We can re-write the stopping rule (4.1) as

$$N_1 = \inf\left[n_1^* \geq m : \sum_{j=1}^{n_1^*-1}\left(\frac{Y_j}{2}\right) \leq n_1^{*^2}\left(n^*\right)^{-1}\right], \quad (4.3)$$

With $Y_j \overset{d}{=} \chi^2_{(2)}$. Let us define a new stopping rule N_1^* by

$$N_1^* = \inf\left[n_1^* \geq m-1 : \sum_{j=1}^{n_1^*}\left(\frac{Y_j}{2}\right) \leq n_1^{*^2}\left(1 + n_1^{*^{-1}}\right)^2\left(n^*\right)^{-1}\right], \quad (4.4)$$

It follows from Swanepoel and Van Wyk (1982) that N_1 and N_1^* follow the same probability distribution. Comparing (4.4) with equation (1.1) of Woodroofe (1977), we obtain in his notations $\alpha = 2,\ \beta = 1,\ \mu = 1,\ \tau = 1,\ \lambda = n^*$,

$L_0 = \left(\frac{3}{2}\right),\ C = \left(n^*\right)^{-1}$ *and* $a = 1$. From Table 2.1 of Woodroofe (1977), $v = .747$. It now follows from Theorem 2.4 of Woodroofe (1977) that, for all $m \geq 3$, *as* $A \to \infty$,

$$E\left(N_1^*\right) = n^* - 2.253 + 0(1).$$

The lemma now follows on using the result that $N_1^* = N_1 - 1$.

Lemma 2: For $\eta \in (0,1)$, *as* $A \to \infty$,

$$P\left(N_1 \leq \eta n^*\right) = O\left(A^{-\frac{(m-1)}{2}}\right).$$

Proof: The result is a direct consequence of Lemma 2.3 of Woodroofe (1977).

Lemma 3: Let the random variable '*W*' be defined by

$$|W - 1| \leq \left|\left(\frac{N_1}{n^*}\right) - 1\right|.$$

Then,

$$W \xrightarrow{a.s.} 1 \text{ as } A \to \infty \quad (4.5)$$

and

W^{-4} is uniformly integrable for all $m \geq 6$. (4.6)

Proof: From (4.1),

$$A^{1/2}\hat{\sigma}_{N_1} \leq N_1 \leq A^{1/2}\hat{\sigma}_{N_1} + (m-1),$$

or,

$$\left(\frac{\hat{\sigma}_{N_1}}{\sigma}\right) \leq \frac{N_1}{n^*} \leq \left(\frac{\hat{\sigma}_{N_1}}{\sigma}\right) + \frac{(m-1)}{n^*}. \quad (4.7)$$

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal,Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

Applying the result that

$\lim_{d\to 0} N_1 = \infty$ *and* $\hat{\sigma}_{N_1} \xrightarrow{a.s.} \sigma$ *as* $N_1 \to \infty$, we obtain from (4.7) that

$$\lim_{A\to\infty}\left(\frac{N_1}{n^*}\right) = 1 \text{ a.s.}$$

Result (4.5) now follows from the definition of N_1.

We note that on the event $(N_1 \le \eta n^*)$,

$$|W-1| \le \left|\left(\frac{N_1}{n^*}\right)-1\right| \le 1-\left(\frac{m}{n^*}\right),$$

i.e. $W^{-1} \le \left(\frac{n^*}{m}\right)$. Thus, denoting by $I(\cdot)$, the usual indicator function and *k*, any positive generic constant independent of *A*, we have

$$E\left[W^{-4} I\left(N_1 \le \eta n^*\right)\right] \le k\left(n^*\right)^4 P\left(N_1 \le \eta n^*\right) \quad (4.8)$$

Applying Lemma 3, we obtain from (4.8) that

$$E\left[W^{-4} I\left(N_1 \le \eta n^*\right)\right] \le k\left(n^*\right)^4 P\left(N_1 \le \eta n^*\right) = O\left(A^{\frac{(m-1)}{2}-2}\right)$$

$$= o(1), \text{ as } A \to \infty, \text{ for all } m \ge 6. \quad (4.9)$$

Furthermore, on the event $(N_1 \ge \eta n^*)$, $W^{-1} \le \eta^{-1}$. Thus,

$$E\left[W^{-2} I\left(N_1 > \eta n^*\right)\right] \le kP\left(N_1 > \eta n^*\right)$$

$$= o(1), \text{ as } A \to \infty, \quad (4.10)$$

Since N_1 terminates with probability one. Result (4.6) now follows on combining (4.9) and (4.10).

Lemma 4: For $r(>0)$, $\left(\frac{N_1}{n^*}\right)^r$ is uniformly integrable.

Proof: See Lemma 2.1 of Woodroofe (1977).

Lemma 5: For $r(>0)$, $\left|\frac{\left(N_1-n^*\right)}{\left(n^*\right)^{1/2}}\right|^r$ is uniformly integrable for all $m > 1+\left(\frac{r}{2}\right)$.

Proof: The result follows from Theorem 2.3 of Woodroofe (1977).

In what follows, we denote by $S_n = \sum_{j=1}^{n}\left(\frac{Y_j}{2}\right)$.

Lemma 6: For all $m \ge 3$,

$\frac{\left(N_1-n^*\right)}{\left(n^*\right)^{1/2}}$ and $\frac{\left(S_{N_1}-N_1\right)^2}{\left(n^*\right)^{1/2}}$ are asymptotically uncorrelated, (4.11)

and

for all $m \ge 4$,

$\frac{\left(N_1-n^*\right)^2}{\left(n^*\right)}$ and $\frac{\left(S_{N_1}-N_1\right)^2}{\left(n^*\right)^{1/2}}$ are asymptotically uncorrelated, (4.12)

Proof: By Cauchy-Schwartz inequality, we have

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

$$Cov^2\left\{\frac{(N_1-n^*)}{(n^*)^{1/2}},\frac{(S_{N_1}-N_1)^2}{(n^*)^{1/2}}\right\}\le Var\left\{\frac{(N_1-n^*)}{(n^*)^{1/2}}\right\}Var\left\{\frac{(S_{N_1}-N_1)^2}{(n^*)^{1/2}}\right\}$$

$$\le E\left\{\frac{(N_1-n^*)^2}{(n^*)}\right\}E\left\{\frac{(S_{N_1}-N_1)^2}{(n^*)}\right\}. \qquad (4.13)$$

It follows from Lemma 5 of Chow and Yu (1981) that $\left\{\frac{(S_{N_1}-N_1)^2}{(n^*)}\right\}$ is uniformly integrable. $\qquad (4.14)$

Application of Lemma 5 and (4.14) lead us to (4.11). A similar proof holds for (4.12).

Lemma 7: $\frac{(N_1-n^*)}{(n^*)^{1/2}} \xrightarrow{L} N(0,1)$, *as* $A\to\infty$.

Proof: The result follows from Bhattacharya and Mallik (1973). One can also follow the technique of the proof of the Theorem 3.

The main result is now proved in the following theorem, which provides second-order approximations for the risk corresponding to the sequential procedure $(1.4.1)$

Theorem 7: For all $m\ge 6$, *as* $A\to\infty$,

$$R_{N_1}(A)=5n^*+10.747+o(1).$$

Proof: We can write (4.2) as

$$R_{N_1}(A)=A\sigma^2E\left\{\frac{(\overline{X}_{N_1}-\mu)}{\sigma}-1\right\}^2+E(N_1)$$

$$=A\sigma^2E\left\{\frac{1}{N_1}\sum_{i=1}^{N_1}\frac{(X_i-\mu)}{\sigma}-1\right\}^2+E(N_1)$$

$$=(n^*)^2E\left[\frac{1}{N_1^2}(S_{N_1}-N_1)^2\right]+E(N_1)$$

$$=E\left[f\left(\frac{N_1}{n^*}\right)(S_{N_1}-N_1)^2\right]+E(N_1),$$

Where $f(x)=x^{-2}$. Expanding $f(x)$ around $(x=1)$ by Taylor's series, we obtain for $|W-1|\le\left|\left(\frac{N_1}{n^*}\right)-1\right|$,

$$R_{N_1}(A)=E\{(S_{N_1}-N_1)^2\}-2E\left\{\left(\frac{N_1}{n^*}-1\right)(S_{N_1}-N_1)^2\right\}+3E\left\{\left(\frac{N_1}{n^*}-1\right)^2W^{-4}(S_{N_1}-N_1)^2\right\}+E(N_1). \qquad (4.15)$$

It follows from Wald's lemma for cumulative sums that

$$E(S_{N_1}-N_1)^2=Var(Y_j)E(N_1)$$

$$=4E(N_1). \qquad (4.16)$$

Applying Lemmas 1, 3, 4, 5, 6, 7, and (4.16), we obtain from (4.15) that, for all $m\ge 6$, *as* $A\to\infty$,

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal, Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

$$R_{N_1}(A) = 5E(N_1) - E\left\{\frac{(N_1 - n^*)}{(n^*)^{1/2}} \cdot \frac{(S_{N_1} - N_1)^2}{(n^*)^{1/2}}\right\} + 3E\left\{\frac{(N_1 - n^*)^2}{(n^*)}, \frac{(S_{N_1} - N_1)^2}{(n^*)} W^{-4}\right\}$$

$$= 5E(N_1) - \frac{4}{n^*} E(N_1 - n^*)E(N_1) + \frac{12}{n^*} E(N_1)$$

$$= 5\{n^* - 1.253 + o(1)\} - \frac{4}{n^*}\{-1.253 + o(1)\}\{n^* - 1.253 + o(1)\} + \frac{12}{n^*}(n^* - 1.253 + o(1))$$

$$= 5n^* + 10.747 + o(1),$$

and the theorem follows.

Remarks 3: The method of obtaining the second-order approximations for the risk presented in Theorem 7 is simpler as compared to that of Woodroofe (1977), as it does not require complicated estimation of various components comprising the risk.

Let us now consider the sequential procedure based on the UMVUE of σ.

Let us take $m(\geq 2)$ to be the initial sample size. Then, the stopping time $N_1 \equiv N_2(A)$ is defined by

$$N_2 = \inf\left[n_2 \geq m : n_2 \geq A^{1/2} \hat{\sigma}^{*^2}_{n_2}\right]. \qquad (4.17)$$

After stopping, we estimate θ by $\overline{X}_{N_2}$, having the associated risk

$$R_{N_2}(A) = AE\left[\left(\overline{X}_{N_2} - \theta\right)^2\right] + E(N_2). \qquad (4.18)$$

Now we state the following theorem, which provides second-order approximations for the risk (4.18).

The proof of the theorem is similar to that of Theorem 8. We omit the details for brevity.

Theorem 9: For all $m \geq 6$, *as* $A \to \infty$,

$$R_{N_2}(A) = 5n^* + 11.747 + o(1).$$

Remark 4: From Theorem 8 and 9, we conclude that the risk corresponding to the stopping rule N_2 is higher than that associated with N_1. Thus, the use of MLE of σ is preferred than its UMVUE.

REFERENCES

[1] Anscombe, F.J. (1952): Large sample theory of sequential estimation. Proc. Cambridge Philos. Soc., 48, 600-607.

[2] Basu, A.P. (1971): On a sequential inspection and statistical decisions. Jour. Roy. Statist. Soc., B16, 151-174.

[3] Bhattacharya, P.K. and Mallik, H. (1973): Asymptotic normality of the stopping times of some sequential procedures. Ann. Statist., 1, 1203-1211.

[4] Chaturvedi, A. and Shukla, P.S. (1990): Sequential point estimation of location parameter of a negative exponential distribution. Jour. Indian Statist. Assoc., 28,41-50.

[5] Chaturvedi, A. (1996): Sequential interval estimation procedures for the mean exponential survival time and reliability function. Microelectron Reliab., 36, 91-96.

[6] Chow, Y.S. and Robbins, H. (1965): On the asymptotic theory of fixed-width sequential confidence intervals for the mean. Ann. Math. Statist., 36, 457-462.

[7] Epstein, B. and Sobel, M. (1953): Life testing. Jour. Amer. Statist. Assoc., 48, 486-502.

[8] Epstein, B. and Sobel, M. (1954): Some theorems relevant to life testing from an exponential distribution. Ann. Math. Statist., 25, 373-381.

[9] Epstein, B. and Sobel, M. (1955): Sequential life tests in the exponential case. Ann. Math. Statist., 26, 82-93.

[10] Epstein, B. (1954): Truncated life tests in the exponential case. Ann. Math. Statist., 25, 555-564.

[11] Epstein, B. (1960): Testing for the validity of the assumption that the underlying distribution of life is exponential. Technometrics, 2, 83-101, 167-183.

[12] Gupta, R. and Bhougal, S.(2006): Estimating the largest location parameter of k exponentialpopulations having unequal and unknown scale parameters. Assam Stat. Rev.,20,2,88-111.

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459 (Online), An ISO 9001:2008 Certified Journal,Volume 3, Special Issue 2, January 2013)
National conference on Machine Intelligence Research and Advancement (NCMIRA, 12), INDIA.

[13] Manisha, P., M.M. Ali and J. Woo(2005): Estimation and testing of P(Y>X) in two parameter exponential distribution. Statistics, 39,5,415-428.

[14] Mukhopadhayay, N. (1974): Sequential estimation of location parameter in exponential distributions. Cal. Statist. Assoc. Bull., 23, 85-95.

[15] Mukhopadhayay, N. and Hilton, G.F. (1986): Two-stage and sequential procedures for estimating the location parameter of a negative exponential distribution. South African Statist. Jour., 20, 117-136.

[16] Starr, N. and Woodroofe, M. (1972): Further remarks on sequential estimation: The exponential case. Ann. Math. Statist., 43, 1147-1154.

[17] Swanepoel, J.W.H. and Van Wyk, J.W.J. (1982): Fixed width confidence intervals for the location parameter of an exponential distribution. Commun. Statist.-Theor. Math., All, 1279-1289.

[18] Woodroofe, M. (1977): Second-order approximations for sequential point and interval estimation. Ann. Statist., 5, 984-995.

[19] Sukhatme, P.V. (1973): Tests of significance for samples of the χ^2 population with two degrees of freedom. Ann. Eugen., 8, 52-56.

[20] Zacks, S. (1971): The Theory of Statistical inference. John Wiley and Sons, New York.