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LU Factorization Method to Solve Linear Programming Problem

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***Abstract*- This paper presents a new approach for the solution of Linear Programming Problems with the help of LU Factorization Method of matrices. This method is based on the fact that a square matrix can be factorized into the product of unit lower triangular matrix and upper triangular matrix. In this method, we get direct solution without iteration. We also show that this method is better than simplex method.**

***Keywords*- L U Factorization Method, Linear Programming Problem, System of Linear Equations, Unit Lower Triangular Matrix, Upper Triangular Matrix.**

I. INTRODUCTION

All linear programming problems are concerned with maximization (or minimization) of a linear function subject to a set of linear constraints. The simplex method of linear programming was developed by Prof. G. B. Dantzig in 1947. He demonstrated how to use an objective function to find the optimal solution from amongst the several feasible solutions to a linear programming problem. Further, the development of computers during the last three decades has made it easy for the simplex method to solve large scale linear programming problems very quickly. However, in 1984, Narendra Karmarkar of AT & T Bell laboratories was developed a new algorithm for solving very large scale linear programming problems. Further, in 1998, this method was modified by P. Kanniappan and K. Thangavel.

In this paper, we use the LU Factorization Method of Matrices for the system of linear inequalities in linear programming problem to solve it. Here the objective function is also to be considered as a constraint which together with linear inequalities forms a system of linear inequalities. By applying LU Factorization Method, we get the solution just after initial iteration.

This paper has five sections: The first section gives knowledge about LU Factorization method, second section introduce its application for the solution of linear programming problem, the third section illustrates this method with a numerical example, the fourth section highlights the simplex method and the last fifth section gives comparison of LU Factorization method with simplex method.

II. L U FACTORIZATION METHOD

One way of solving a system of linear equations is using the Gauss-Jordan method. Another way of solving a system of linear equations is by using a factorization technique for matrices called LU Factorization. This factorization involves two matrices, one unit lower triangular matrix L and one upper triangular matrix U.

Steps to solve a system of linear equations using an LU decomposition:

1. Set up the system of n linear equations in n variables $x_1, x_2, \dots, x_n$ as a matrix equation $AX = B$, where $A = [a_{ij}]$ is an $n \times n$ matrix of real coefficients, $X = [x_i]$ is $n \times 1$ matrix of variables and $B = [b_i]$ is $n \times 1$ matrix of constants.
2. Find the unit lower triangular matrix L and the upper triangular matrix U such that $LU = A$. This will yield the equation $(LU)X = B$.
3. Let $Y = UX$. Then solve the equation $LY = B$ for Y.
4. Take the values for Y and solve the equation $Y = UX$ for X. This will give the solution to the system $AX = B$.

III. APPLICATION OF LU FACTORIZATION TO SOLVE LINEAR PROGRAMMING PROBLEM

Let us consider the linear programming problem:

Maximize: $z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_{n-1}x_{n-1}$

Subject to the constraints:

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2,n-1}x_{n-1} \le b_2,$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3,n-1}x_{n-1} \le b_3,$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{n,n-1}x_{n-1} \le b_n$$
$$x_1, x_2, x_3, \dots, x_{n-1} \ge 0.$$

We write this L. P. P. in the form of less than inequalities:

To find: z

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Subject to:

$$-c_1x_1 - c_2x_2 - c_3x_3 - \cdots - c_{n-1}x_{n-1} + z \le 0,$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \le b_2,$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3,n-1}x_{n-1} \le b_3,$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{n,n-1}x_{n-1} \le b_n$$
$$-x_1, -x_2, -x_3, \dots, -x_{n-1}, -z \le 0.$$

Now, we can consider the system of linear equations $AX = B$,

where,

$$A = \begin{bmatrix} a_{11} & a_{12} & & a_{1,n-1} & 1 \\ a_{21} & a_{22} & & a_{2,n-1} & 0 \\ a_{31} & a_{32} & \cdots & a_{3,n-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & & a_{n,n-1} & 0 \end{bmatrix},$$
$$a_{1i} = -c_i \text{ for } 1 \le i \le n-1;$$
$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}.$$

In this way, the objective function is considered as a constraint and z is considered as a variable.

A. Case I

If the number of inequalities is equal to the number of variables:

Then LU Factorization method is applied to the system of linear equations $AX = B$. Firstly, we get the matrix Y as an initial iteration and then the matrix X as final iteration.

B. Case II

If the number of inequalities is less than the number of variables:

Add the inequalities in the system till number of inequalities equals the number of variables. We can add the inequalities in the system as below:

Consider the first constraint in given L. P. P.:

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2,n-1}x_{n-1} \le b_2$$

Choose the non zero coefficient in this inequality $a_{2j} \ne 0$ and add the inequality $a_{21}x_j \le b_2$ in the system. Continuing in this way till number of inequalities reaches to the number of variables.

C. Case III

If the number of variables is less than the number of inequalities:

If there are less number of variables than that of inequalities, then we introduce that much number of slack variables in the appropriate inequalities and add $+1$ on R. H. S. of each of that inequalities.

D. Case IV

If there is a zero row in upper triangular matrix U, then the given problem has an infeasible solution and we can stop the process.

IV. ILLUSTRATION

A. Case I

Maximize: $z = 3x_1 + 5x_2 + 4x_3$

Subject to the constraints:

$$2x_1 + 3x_2 \le 8,\ 2x_2 + 5x_3 \le 10,\ 3x_1 + 2x_2 + 4x_3 \le 15,$$
$$x_1, x_2, x_3 \ge 0.$$

Solution by Factorization Method: Let us write the L. P. P. as:

$$-3x_1 - 5x_2 - 4x_3 + z \le 0,$$
$$2x_1 + 3x_2 \le 8,\ 2x_2 + 5x_3 \le 10,\ 3x_1 + 2x_2 + 4x_3 \le 15,$$
$$-x_1, -x_2, -x_3 \le 0.$$

$$\text{Let } A = \begin{bmatrix} -3 & -5 & -4 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 3 & 2 & 4 & 0 \end{bmatrix}.$$

Let $A = LU$, where L is unit lower triangular matrix and U is an upper triangular matrix.

$$\text{This means that } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix},$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}, \text{ and}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} = \begin{bmatrix} -3 & -5 & -4 & 1 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 3 & 2 & 4 & 0 \end{bmatrix}.$$

Therefore, on simplification, we get

$$u_{11} = -3, u_{12} = -5, u_{13} = -4, u_{14} = 1, l_{21} = -\frac{2}{3},$$

$$u_{22} = -\frac{1}{3}, u_{23} = -\frac{8}{3}, u_{24} = \frac{2}{3}, l_{31} = 0,$$

$$l_{32} = -6, u_{33} = -11, u_{34} = 4, l_{41} = -1, l_{42} = 9,$$

$$l_{43} = -\frac{24}{11}, u_{44} = 41/11.$$

Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2/3 & 1 & 0 & 0 \\ 0 & -6 & 1 & 0 \\ -1 & 9 & -24/11 & 1 \end{bmatrix}$,

$$U = \begin{bmatrix} -3 & -5 & -4 & 1 \\ 0 & -1/3 & -8/3 & 2/3 \\ 0 & 0 & -11 & 4 \\ 0 & 0 & 0 & 41/11 \end{bmatrix}.$$

Now Consider $LY = B$, where

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 8 \\ 10 \\ 15 \end{bmatrix}.$$

Then, on simplification, we get initial iteration:

$$y_1 = 0, y_2 = 8, y_3 = 58, y_4 = \frac{765}{11}.$$

Finally, the solution matrix

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ z \end{bmatrix}$$

is given by $UX = Y$.

Hence, on simplification, we get final iteration:

$$x_1 = \frac{89}{41}, x_2 = \frac{50}{41}, x_3 = \frac{62}{41}, z = \frac{765}{41}.$$

This solution obtained by factorization method is same as that by simplex method, but easier to find.

B. Case II

Maximize: $z = 2x_1 + 3x_2$

Subject to the constraints: $x_1 + x_2 \le 1; x_1, x_2 \ge 0$.

Here the system of linear inequalities is

$$-2x_1 - 3x_2 + z \le 0,$$

$$x_1 + x_2 \le 1,$$

$$x_2 \le 1,$$

$$-x_1, -x_2 \le 0.$$

Hence, we have

$$A = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 1 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

That is $L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, U = \begin{bmatrix} -2 & -3 & 1 \\ 0 & -1/2 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$.

Also $B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

Hence the initial iteration is $Y = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ and final iteration is $X = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$.

That is solution is $x_1 = 0, x_2 = 1, z = 3$.

C. Case III

Maximize: $z = 2x_1 + 3x_2$

Subject to the constraints:

$x_1 + x_2 \le 1, 6x_1 + 2x_2 \le 3,\ 2x_1 + 6x_2 \le 3; x_1, x_2 \ge 0$.

Here the system of linear inequalities is

$$-2x_1 - 3x_2 + z \le 0,$$

$$x_1 + x_2 + x_3 \le 2, \qquad 6x_1 + 2x_2 \le 3,$$

$$2x_1 + 6x_2 \le 3;\ -x_1, -x_2, -x_3 \le 0.$$

Hence, we have

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$$A=\begin{bmatrix}-2 & -3 & 1 & 0\\ 1 & 1 & 0 & 1\\ 6 & 2 & 0 & 0\\ 2 & 6 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & 0 & 0\\ -1/2 & 1 & 0 & 0\\ -3 & 14 & 1 & 0\\ -1 & -6 & -1 & 1\end{bmatrix}\begin{bmatrix}-2 & -3 & 1 & 0\\ 0 & -1/2 & 1/2 & 1\\ 0 & 0 & -4 & -14\\ 0 & 0 & 0 & -8\end{bmatrix}$$

That is $L=\begin{bmatrix}1 & 0 & 0 & 0\\ -\frac{1}{2} & 1 & 0 & 0\\ -3 & 14 & 1 & 0\\ -1 & -6 & -1 & 1\end{bmatrix}$,

$$U=\begin{bmatrix}-2 & -3 & 1 & 0\\ 0 & -1/2 & 1/2 & 1\\ 0 & 0 & -4 & -14\\ 0 & 0 & 0 & -8\end{bmatrix}.$$

Also $B=\begin{bmatrix}0\\2\\3\\3\end{bmatrix}$.

Hence the initial iteration is $Y=\begin{bmatrix}0\\2\\-25\\-10\end{bmatrix}$ and final iteration is $X=\begin{bmatrix}3/8\\3/8\\15/8\\5/4\end{bmatrix}$.

Thus the solution is $x_1=\frac{3}{8}, x_2=\frac{3}{8}, z=\frac{15}{8}$.

V. Solution By Simplex Method

By introducing the slack variables $x_4 \geq 0$, $x_5 \geq 0$ and $x_6 \geq 0$, the set of inequalities representing the constraints of the given problem are converted into the set of equations:

$$\begin{bmatrix}2 & 3 & 0 & 1 & 0 & 0\\ 0 & 2 & 5 & 0 & 1 & 0\\ 3 & 2 & 4 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}8\\10\\15\end{bmatrix}.$$

The basic variables are x_4, x_5, x_6. The iterative simplex tables are:

TABLE I
INITIAL ITERATION

c_B	y_B	x_B	y_1	$\boldsymbol{y_2}$	y_3	y_4	y_5	y_6
0	$\boldsymbol{y_4}$	8	2	**3**	0	1	0	0
0	y_5	10	0	2	5	0	1	0
0	y_6	15	3	2	4	0	0	1
	z	0	-3	-5	-4	0	0	0

Introduce y_2 and drop y_4.

TABLE II
FIRST ITERATION

c_B	y_B	x_B	y_1	y_2	$\boldsymbol{y_3}$	y_4	y_5	y_6
5	y_2	8/3	2/3	1	0	1/3	0	0
0	$\boldsymbol{y_5}$	14/3	-4/3	0	**5**	-2/3	1	0
0	y_6	29/3	5/3	0	4	-2/3	0	1
	z	40/3	1/3	0	-4	5/3	0	0

Introduce y_3 and drop y_5.

TABLE III
SECOND ITERATION

c_B	y_B	x_B	$\boldsymbol{y_1}$	y_2	y_3	y_4	y_5	y_6
5	y_2	8/3	2/3	1	0	1/3	0	0
4	y_3	14/15	-4/15	0	1	-2/15	1/5	0
0	$\boldsymbol{y_6}$	89/15	**41/15**	0	0	-2/15	-4/5	1
	z	256/15	-11/15	0	0	17/15	4/5	0

Introduce y_1 and drop y_6.

TABLE IV
THIRD ITERATION

c_B	y_B	x_B	y_1	y_2	y_3	y_4	y_5	y_6
5	y_2	50/41	0	1	0	1/3	0	0
4	y_3	62/41	0	0	1	-2/15	1/5	0
11/15	y_1	89/41	1	0	0	-2/41	-12/41	15/41
	z	256/15	-11/15	0	0	17/15	4/5	0

Now since all $z_j - c_j$ are non-negative, therefore an optimum basic feasible solution is

$$x_1=\frac{89}{41}, x_2=\frac{50}{41}, x_3=\frac{62}{41}; \text{max. z}=\frac{765}{41}.$$

VI. Comparison Of Simplex Method With LU Factorization Method

A. *Similarities*

(1) Both methods are iterative methods.
(2) Both methods gives us an actual solution.

B. *Differences*

(1) In simplex method, we always introduce slack variables. In LU factorization method, we introduce the slack variables only in case III.

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(2) In simplex method, we have to take at least two iterations. In LU factorization method, we have to take only two iterations.

VI. Conclusions And Remarks

LU factorization method has fewer calculations as compare to the simplex method. LU factorization method has simple calculations as compare to the simplex method. After practice, LU factorization method becomes very mechanical. So this method is as faster as that of the simplex method.

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