Robustness Analysis of Optimal Regulator for Vehicle Model with Nonlinear Friction

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Abstract—This paper derive an evaluation index of the robustness of nonlinear optimal regulators designed by the stable manifold method with respect to unexpected perturbations, i.e., observation noises or input disturbances in a vehicle model with nonlinear friction from a geometric distance between disturbed states and an optimal state. The validity of the indexes are shown by a numerical example.

Keywords—Hamilton-Jacobi equations, nonlinear systems, optimal controls, robustness, stable manifold method

I. INTRODUCTION

Nonlinear optimal control designs can be described as a problem of finding stabilizing solutions of Hamilton-Jacobi equations. Solving Hamilton-Jacobi equations had been considered to be difficult for a long time; however, an effective numerical solver called the stable manifold method has been recently proposed [1]. The stable manifold method has been applied to various control problems [2]. Stable manifolds obtained from stabilizing solutions of Hamilton-Jacobi equations consist of the pairs of optimal orbits converging to the origin and optimal gains that are associated to the orbits at each point of time. That is, the stable manifolds can be used as a scheduled gain map in online controls. On the other hand, when disturbances are applied to the systems, state variables may be translated to some other position outside of the stable manifolds. Hence, the scheduled gains are no longer optimal, but also they might cause instability in the worst case. Then, optimal gains calculated from a stable manifold prepared in an offline process become mismatched gains. Robustness for such a system variation has not been sufficiently discussed.

In this paper, we study the robustness of the nonlinear optimal regulators, themselves. Although the $H^\infty$ controller [3] can be designed by the stable manifold method [1], the control is effective only for behaviors around a stationary state. We propose an evaluation index of the robustness from the distance of state variables with mismatched gains from stable manifolds.

II. SUMMARY OF STABLE MANIFOLD METHOD

A. Nonlinear optimal control problems

Nonlinear optimal controllers are designed by finding stabilizing solutions of Hamilton-Jacobi equations. In this paper, we focus our attention on the design of nonlinear optimal regulators.

Let us consider the input affine nonlinear system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t),$$  \hspace{1cm} (1)

where $x(t) \in \mathbb{R}^n$ is the stage variable, $u(t) \in \mathbb{R}^m$ is the control input, $f, g$ are smooth vector fields, and we assume that $f(0) = 0$. Then, we consider the following optimal problem.

Definition 1. (Nonlinear optimal control problem) For the system (1), find a control input $u = u_{opt}$ that minimizes the following cost function $J$:

$$J = \int_0^\infty \{q(x(t)) + u^T(t)R u(t)\} \, dt,$$  \hspace{1cm} (2)

where $q: \mathbb{R}^n \to \mathbb{R}$ is a semi-positive definite function such that $q(0) = 0$ and $(\partial q/\partial x)(0) = 0$, and $R$ is a positive definite matrix.

The necessary condition of the above problem can be written as a Hamilton-Jacobi equation or a Hamiltonian system. Here, we assume that optimal control inputs exist. We define the minimum value $V$ of the cost function (2) for some initial condition as follows:

$$V(x(t), t) = \min_{u(t)} \int_t^\infty \{q(x(\tau)) + u^T(\tau) R u(\tau)\} \, d\tau,$$  \hspace{1cm} (3)

where $V$ called a value function doesn't depend on $t$ explicitly because of the time invariance of the system (1) and the infinite terminal time of the cost function (2). Thus, $V(x(t), t) = V(x(t))$. 


By defining the Hamiltonian
\[ H = q + u^T R u + \left( \frac{\partial V}{\partial x} \right)^T (f + g u), \]  
(4)

The following necessary condition of the optimal control is derived from dynamic programming:
\[ 0 = \min_u H \left( x, u, \frac{\partial V}{\partial x} \right). \]  
(5)

From the condition \( \partial H/\partial u = 0 \), the optimal control input can be given by
\[ u_{opt} = -\frac{1}{2} R^{-1} g^T \frac{\partial V}{\partial x}. \]  
(6)

By substituting \( u_{opt} \) into (4), we get the Hamilton-Jacobi equation
\[ H = p^T f - \frac{1}{2} p^T R p + q = 0, \]  
(7)

where \( p = \partial V/\partial x \) and \( \bar{R}(x) = g(x)R^{-1}g^T(x) \).

Equation (7) can be transformed into the Hamiltonian system
\[ \begin{cases} \dot{x} = f - \frac{1}{2} \bar{R} p, \\ \dot{p} = -\left( \frac{\partial f}{\partial x} \right)^T p + \frac{1}{4} \frac{\partial (p^T \bar{R} p)}{\partial x} - \left( \frac{\partial q}{\partial x} \right)^T. \end{cases} \]  
(8)

We define solutions of the equation (7) that ensure the asymptotic stability of the closed loop systems: \( \lim_{t \to \infty} x(t) = 0 \) as follows.

**Definition 2.** A solution of Hamilton-Jacobi equation (7) is called a stabilizing solution if the solutions satisfy
\[ \frac{\partial q}{\partial x} = 0 \text{ and } f(x) = 0. \]

The linearized relation of the Hamilton-Jacobi equation (7) at the origin is equivalent to the Riccati equation
\[ PA + A^T P - PBR^{-1}B^T P + Q = 0, \]  
(9)

Where \( \partial^2 V/\partial x^2 = P \), the matrices \( A, B, Q \) are defined by \( f(x) = Ax + O(|x|^2) \), \( B = g(0) \) and \( q(x) = (1/2)x^T Q x + O(|x|^3) \), respectively.

**Definition 3.** The symmetric matrix \( P \) is called a stabilizing solution if \( P \) satisfies the Riccati equation (9) and \( A - B R^{-1}B^T P \) is a stable matrix.

On the existence of solutions of Hamilton-Jacobi equations, the following fact is known [1], [4].

**Theorem 4.** If there exists a stabilizing solution \( P \) in the Hamilton-Jacobi equation (7) linearized at the origin, i.e., the Riccati equation, then the Hamilton-Jacobi equation (7) has a stabilizing solution in the neighborhood of the origin.

**B. Stable manifold method**

The partial derivative of a stabilizing solution of the Hamilton-Jacobi equation (7), i.e., \( \partial V/\partial x \) can be described as the stable manifold of the Hamiltonian system (8).

The stable manifold can be calculated by the stable manifold method [1].

The following procedure gives required information for an iterative calculation of stable manifolds:

1) Calculate a symmetric matrix \( P \) that is equal to a stabilizing solution of equation (9) obtained from equation (7) that is linearized at the origin.
2) Calculate a matrix \( S \) that is a solution of Lyapunov equation \( FS + SF^T = F \), where \( F = (A - \bar{R}(0)P) \).
3) By applying the coordinate transformation
\[ \begin{bmatrix} x' \\ p' \end{bmatrix} = T^{-1} \begin{bmatrix} x \\ p \end{bmatrix}, \quad T = \begin{bmatrix} I & S \\ P & PS + I \end{bmatrix} \]  
(10)

To system (8), the following representation is given:
\[ \begin{bmatrix} x' \\ p' \end{bmatrix} = \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix} \begin{bmatrix} x' \\ p' \end{bmatrix} + \begin{bmatrix} n_x \\ n_u \end{bmatrix} \]  
(11)

Where \( n_x = n_x(t, x', p') \) and \( n_u = n_u(t, x', p') \) are higher order derivatives.

A solution \( (x, p) \)-space transformed from that of (11) in \( (x', p') \)-space by \( T \) consists of an optimal trajectory \( x \) and an optimal feedback gain \( p \).

The stable manifold of (8) can be calculated by the following iteration:

1) We rewrite equation (11) as the following 2n-dimensional differential equation:
\[ \begin{cases} \dot{x} = Fx + n_x(t, x, p), \\ \dot{y} = -F^T y + n_u(t, x, p). \end{cases} \]  
(12)

2) Calculate sequences of functions \( \{x_k(t, \xi)\} \) and \( \{y_k(t, \xi)\} \) defined by
\[ x_{k+1}(t, \xi) = e^{Ft} x_k(t, \xi) \]
\[ + \int_0^t e^{F(t-s)} n_x(s, x_k(s, \xi), y_k(s, \xi)) \, ds, \]
\[ y_{k+1}(t, \xi) = \]
\[ - \int_t^\infty e^{-F(t-s)} n_u(s, x_k(s, \xi), y_k(s, \xi)) \, ds, \]  
(13)
For some parameter $\xi \in \mathbb{R}^n$, where $x_0(t, \xi) = e^{P(t)} \xi, y_0(t, \xi) = 0$.

3) Choose an initial vector in a plain surface defined by $P$.

4) Extend a solution along $\xi$ by the iterative calculation of (13).

5) Check whether the Hamiltonian of the right side of (7) is sufficiently close to zero. After a sufficient convergence, the solution can be regarded as the Hamilton-Jacobi equation (7).

6) If the solution passes through a desired initial state of the control system, then the iteration is finished. If not, back to 3).

The solutions far away from the origin are different from the plain surface of stabilizing solutions of the Riccati equation. The difference means a nonlinear optimal feedback gain.

III. MAIN RESULTS

A. Problem setting and mismatched feedback gains

A solution of the Hamilton-Jacobi equation (7) consists of the pair of optimal gains $p(t)$ and optimal orbits $x(t)$. Hence, the stable manifolds are used as a scheduled gain map in the online process of the controls. The gradient of the stable manifolds in the neighborhood of a nominal orbit at each time represents a robustness of the systems. Because, when the system state is translated by some perturbation, a mismatched gain applied to the systems is different from the optimal gain if the perturbation cannot be detected or control inputs have time delay. Such differences might make the systems unstable in the worst case.

**Proposition 5.** Let $H(x, p) = 0$ be a stable manifold describing a set of stabilizing solutions derived from some nonlinear optimal control problem for the system (1). Consider a perturbation: $x(t) \rightarrow x'(t)$ to the system at a certain time $t$. Then, the orbits get away from the stable manifold except in the case of the perturbed state consisting of position and velocity is just the same of some optimal state. The mismatched gain $p(t)$ for $x(t)$ that is different from the optimal gain $p'(t)$ for $x'(t)$ can cause the unstability of the controlled system.

**Proof:** The effect of the mismatched gains can be described as an additional term in the right side of the first equation of (8). Such a feedback term can change the system to be unstable.

**Remark 6.** Practically, ‘small perturbations’ (see below) in the systems don’t always cause serious problems in stabilization. The stable manifold has a foliation because of Hamiltonian systems. Hence, the neighborhoods of orbits on the stable manifold are densely covered by unstable orbits to the origin. Once a state is translated out of optimal orbits, it is on unstable orbits in general. However, perturbed systems with states on unstable orbits can be stabilized if they sufficiently reach in the neighborhood of the origin by linear regulators. Therefore, ‘small perturbations’ means to translate a state to orbits that reach to the neighborhood within a finite time.

B. Robustness with respect to disturbances

We shall introduce the following measure of the robustness from the fact in Proposition 5.

**Proposition 7.** Let $H: \mathcal{E} \rightarrow \mathbb{R}$ be the Hamiltonian of the Hamilton-Jacobi equation (7), where the natural projection $\pi: \mathcal{E} \rightarrow \mathcal{X}; (p, x) \mapsto x$ is the bundle consisting of the total space $\mathcal{E}$, the base space $\mathcal{X}$ and the fiber $\mathcal{P}_x = \pi^{-1}(x)$ on each $x \in \mathcal{X}$. Let $v = v^i(\partial / \partial x^i) \in T_x \mathcal{X}$ be a unit vector field normalized with respect to time at each $x \in \mathcal{X}$. Then, the pairing $(\cdot, \cdot): T^*_x \mathcal{X} \times T_x \mathcal{X} \rightarrow \mathbb{R}$ between $dH \in \Omega^1(\mathcal{P})$ restricted to $\mathcal{X}$ and $dv \in T_x \mathcal{X}$ can be defined by

$$\frac{\partial H}{\partial v} = (dH|_x, v) = -\dot{p}_i v^i, \quad (14)$$

where $|_x$ represents the restriction to $\mathcal{X}$, we have defined

$$v = v(t) = v^i(t) \frac{\partial}{\partial x^i}. \quad (15)$$

The coefficient $v^i = v^i(t)$ is a function, we have used the relation

$$dx^i \cdot \frac{\partial}{\partial x^i} = \delta_{ij}, \quad (16)$$

And $\delta_{ij}$ is the Kronecker delta.

**Proof:** From the direct calculation, we obtain

$$dH = \dot{x} \cdot dp - \dot{p} \cdot dx, \quad (17)$$

$$\langle dH|_x, v \rangle = - \sum_i p_i dx^i \cdot v^i \frac{\partial}{\partial x^i} = -\dot{p}_i v^i. \quad (18)$$

Thus, we can calculate (14). \( \blacksquare \)

**Remark 8.** Equation (14) can be considered as a number of contour lines defined by the 1-form $dH|_x$ that the vector $v$ passes through.
If the perturbation can be detected, optimal gains for the disturbed systems can be applied in theory. However, there exist time delays in control inputs, and the gain mismatch might arise from numerical errors. Hence, we propose the following robustness in such a case.

**Definition 9.** We define the gradient of gains $p(t)$ with respect to a direction spanned by a vector field $v(t)$ along an optimal orbit $x(t)$ as follows:

$$\frac{\partial p}{\partial v} = \langle dp|_x, v \rangle = \sum_{i} \frac{\partial p_i}{\partial x^i} dx^i \cdot v^i \frac{\partial}{\partial x^i} = \frac{\partial p_i}{\partial x^i} v^i.$$  \hspace{1cm} (19)

In an area of stable manifolds in which the gradient of gains is steep, the robustness is low.

**Remark 10.** The action of $df \in T_x^*X$ to $v \in T_xX$ is written as

$$\langle df, v \rangle = vf.$$  \hspace{1cm} (20)

This action can be regarded as a directional derivative, which we denoted by $\partial f/\partial v$. Let $f(x)$ be a function defined on $X$. The ratio of variations of $f(x)$ along a curve $\phi(t)$ is described by

$$\frac{df(\phi(t))}{dt} = \sum_i \frac{\partial f}{\partial x^i} \frac{dx^i(\phi(t))}{dt} = \nabla f,$$  \hspace{1cm} (21)

where we have defined the differential operator

$$\nabla := \frac{dx^i(\phi(t))}{dt} \frac{\partial}{\partial x^i}.$$  \hspace{1cm} (22)

That means a tangent vector at each point $\phi$ to the direction given by $\phi(t)$. That is, tangent vectors to a local coordinate axis is obtained from

$$\nabla x^i = \sum_j \frac{\partial x^i}{\partial x^j} \frac{dx^j}{dt} = \frac{dx^i}{dt}.$$  \hspace{1cm} (23)

Hence, we interpret $\nabla$ as $v$ in our problem. In actual control designs, we evaluate the pairing (14) or (19) along an optimal orbit with respect to directions to which disturbances might be applied. Then, the vector field $v$ at each time must be taken as a unit vector scaled by each control period.

**C. Robustness to disturbances in the worst case**

From the above definitions of robustness, we can introduce an evaluation index of robustness in the worst case in the same sense of $L^2$-gain in nonlinear robust controls [4].

**Definition 11.** Let $z \in \{H, p\}$ be the reference output. Let $w$ be the set of unit vector fields describing the directions of disturbances at each point of optimal orbits (w is equivalent to $v$ in the previous discussion). Then, we define

$$\sup_w \frac{\partial z}{\partial w}.$$

This index describes the set of the weakest directions to disturbances at each point of an optimal orbit.

**IV. NUMERICAL EXAMPLE**

We demonstrate the validity of the two robust indexes defined by the pairings (14) and (19) discussed in the previous section by means of a control example.

**A. Control model**

The purpose of controls is to stabilize a vehicle model [5] to some direction under a constant speed. We assume that the properties of the left side and right side wheels are same. Then, the following equivalent 2-wheel model with respect to yawing can be given by removing rolling and pitching:

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{\theta} \\ \dot{\delta} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\sin \beta}{mV_0} F_x + \frac{\cos \beta}{mV_0} F_y - r \\ 2l_f C_f \cos \delta - \frac{2l_r}{l} C_r \\ r \\ 0 \\ V_0 \sin(\beta + \theta) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u + w,$$  \hspace{1cm} (25)

where $u$ is the control input, $w$ is a disturbance, $\beta$ is the slip angle at the center of gravity (COG), $\delta$ is the slip angle of wheels, $C_f$ is the lateral force of wheels, $i \in \{f, r\}$ means the front and the rear wheels, respectively, $r$ is the yaw rate, $\theta$ is the direction, $\delta$ is the steering angle, and the pair $(X, Y)$ means the position of the vehicle; however, we don’t control the variable $X$ under assuming steady motions (see Fig.1).
In (25), we have used the parameters: the constant speed $V_0 = 17.7$, the mass $m = 990$, the moment of inertia $I = 683$, the distance from front axle to COG $l_f = 1.0$, and the distance from rear axle to COG $l_r = 1.3$. Furthermore, the translational forces $F_x$ and $F_y$, and the cornering force of each wheel $Y_i$ are written as follows:

$$F_x = 2Y_f \sin(\beta_f + \delta) + 2Y_r \sin \beta_r,$$  
$$F_y = 2Y_f \cos(\beta_f + \delta) + 2Y_r \cos \beta_r,$$  
$$Y_i = C_i \cos \beta_i$$

For $i \in \{f, r\}$, and the relation between $C_i$ and $\beta$ can be expressed as follows:

$$C_i = \mu N_i \sin[a \arctan(b \beta_i - c (b \beta_i) - \arctan(b \beta_i))].$$  

Where $a, b, c$ are experimental parameters and $\mu$ is an friction constant between road surface and tire, and $N_i$ is vertical load of each wheel. In this simulation, we set $a = 1.23$, $b = 3.25$, $c = -6.00$, $N_f = 5.48$, $N_r = 4.21$ and $\mu = 0.2$.

In the cost function (2), the weight matrixes $Q, R$ are

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R = 0.5.$$  

Where $q(x) = (1/2)x^T Q x$.

**B. Numerical results**

We consider the following side wind effect $w\cdot I$ as a disturbances in (25):

$$w = \begin{bmatrix} \cos(\beta + \theta) \\ \frac{mV_0}{w_1} \cos(\beta + \theta) \end{bmatrix}.$$

Fig.2 shows the absolute value of the pairing (14), i.e., that of the exterior derivative of the Hamiltonian along the optimal orbit of the system (25) illustrated in Fig.3. At a point of the time coordinate which a value of the pairing is large, the Hamiltonian deviates from zero if a disturbance is applied to the system; therefore, the system tends to be unstable. The variation of the stable manifold is steep at such a point. Indeed, Fig.4 shows that the system is stable to $w_1$ and robust during $3.4 \leq t \leq 3.5$. On the other hand, Fig.5 shows that the system is unstable to $w_1$ for $0.1 \leq t \leq 0.2$ and the robustness is low.

Fig.6 shows the absolute value of the pairing (19), i.e., that of the exterior derivative of the optimal gain along the optimal orbit of the system (25) illustrated in Fig.3. In the following tests, we assumed that the perturbation can be detected. Hence, we applied optimal gains for the disturbed systems. Indeed, Fig.7 shows that the system is stable to $w_1$ and robust during $1.7 \leq t \leq 1.8$. However, Fig.8 shows that the system is unstable to $w_1$ for the low-robust area in $0.4 \leq t \leq 0.5$.

The curves in Fig.9 express the values of the pairing (14) with respect to directions spanning the neighborhood of each point of the optimal orbit at regular intervals. Hence, the maximum of the curves means the evaluation index of robustness in the worst case, i.e., (24).
We proposed an evaluation index of the robustness of nonlinear optimal regulators with respect to unexpected perturbations in terms of the stable manifold method, and showed the validity of the index by a numerical example.

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