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Investigation of Dispersion Characteristics of Gyroelectric Medium Loaded Metallic Waveguide in the Neighbourhood of the Branch Points by Aid of Algebraic Function Approximation

Ersoy Kelebekler[1], Namık Yener[2]
[1]*Electricity and Energy Dep. Uzunciftlik Nuh Cimento VHS of Kocaeli University, Kocaeli, Turkey*
[2]*Department of Biomedical Engineering, Kocaeli University, Kocaeli, Turkey*

***Abstract*— In the study, the branch points on dispersion curves and the dispersion characteristics in the neighborhood of the branch points for gyro-electric medium loaded lossless cylindrical metallic waveguide are investigated by expressing the propagation constant in the form of infinite series by aid of algebraic function theory. The presented method uses the linear algebraic equations system of transmission line voltages and currents belonging to the structure and the algebraic characteristic equation of the truncated version of this system to obtain coefficients of the series. This approximation provides a model of the dispersion curves in the neighborhood of the branch points with Puiseux series which has only two non-zero coefficients and no negative power terms. In numerical examples, dispersion characteristics of plasma column loaded cylindrical metallic waveguide are obtained from the exact solution, the Method of Moment and the Puiseux series and the results are presented comparatively on the same figures.**

***Keywords*— Branch point, metallic closed waveguide, algebraic function theory, method of moment, Puiseux series expansion.**

I. INTRODUCTION

As known, Maxwell equations do not permit an the explicit solution for all physical structures. In certain problems however, semi-analytical solutions can be obtained by expressing the fields in terms of a series expanded in a complete set of functions. The general name of this method which is thus based on reducing Maxwell's differential equations to a linear algebraic system of equations is the Method of Moment (MoM).

With the ever growing power of electronic computers it has become more desirable to solve complex electromagnetic problems using analytic methods that are simple but supported by voluminous computations. In fact problems of arbitrary geometry and physical structure can be solved by computer programming.

MoM is one of such computational methods that is versatile and can give more accurate results if the number of expansion functions is enlarged. Another advantage of the method is its usability for geometrically and physically complex structures [1].

In our study, "Generalized telegrahist's equations" of Schelkunoff [2] have been utilized to produce linear algebraic equations system of plasma column loaded cylindrical waveguide which is taken up as the gyro-electric medium loaded lossless metallic waveguide. In his study, Schelkunoff modeled the waveguide as a system of infinite transmission lines. As the consequence of modeling the waveguide as a system of infinite transmission lines, unknown fields are represented as series of empty guide fields with unknown coefficients of currents and voltages. Thus Maxwell's partial differential equations transform into ordinary differential equations which consist of derivatives with respect to only propagation direction variable. The ordinary differential equations system transforms into a linear equations system because the derivative with respect to propagation direction variable corresponds to multiplication with propagation constant because of the assumed $e^{-\gamma z}$ dependence of the fields where z is the propagation direction coordinate. As a result, the problem transforms into an eigenvalue problem. The coefficients matrix of acquired system contains series impedance and parallel admittance matrices per unit length and its elements are obtained by calculating double-decked integrals on crosswise section of waveguide.

The eigenvalues of the linear algebraic equation system are equal to propagation constants. In this method, field expressions consist of infinite series whose coefficients are current and voltage terms. The method, which uses known analytic solutions of the empty waveguide as base functions, is a semi analytic method due to truncation of the series at a certain point.

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The method is also Galerkin version of MoM because its base and test functions are the same [3-5].

The electromagnetic problem is solved by expanding the fields of the wave guide in Fourier series where the expansions functions are the TE and TM mode fields of the same waveguide as under study, but which is empty. Thus the expanded fields are warranted to satisfy the boundary conditions on the periphery of the waveguide. The coefficients of the said Fourier series are voltage and current variables for each TE and TM mode of the empty waveguide. These coefficients are interrelated by taking the inner products of the unknown fields that satisfy Maxwell's equations by the TE and Tm mode fields of the empty waveguide which are orthogonal to each other.

If we restrict the loading class of waveguide under study by a heterogeneous class in which transverse and longitudinal field components do not couple (i.e. loading medium is gyrotropic). as a result of MoM, the linear algebraic equation system is obtained as below.

$$\begin{bmatrix} \gamma(p)v(p) \\ \gamma(p)i(p) \end{bmatrix} = \begin{bmatrix} 0 & Z(p) \\ Y(p) & 0 \end{bmatrix} \begin{bmatrix} v(p) \\ i(p) \end{bmatrix} \quad (1)$$

where, $\gamma(p)=\alpha+j\beta$ and $p=\sigma+j\omega$ are complex propagation constant and complex frequency, respectively. $Z(p)$ and $Y(p)$ represent series impedance and parallel admittance per unit length matrices which are calculated with double decked integrations of Maxwell's equations on the waveguide cross-section due to the mentioned inner products. Besides, $v(p)$ and $i(p)$ show vectors of the said voltage and current variables and which at the same time are the voltage and current variables on the corresponding transmission line model of the waveguide.

Dimension of transmission line equations system given in Equation (1) is determined with number of TE and TM modes (or eigenfunctions) of empty guide. If number of eigenfunctions used for solution are m, dimension of the system is (mxm). As mentioned above, dimension of the equation system must be infinite to obtain the exact characteristics of the real physical problem. But an approximation is done to the problem, due to obligation of constraint on system dimension at a finite value.

From (1) it is evident that the square of the propagation constant γ^2 is the eigenvalue of the product matrix of $Z(p)$ and $Y(p)$. So the problem under study is thus transformed into an eigenvalue problem. Furthermore because of Eq (1), $Z(p)Y(p)$ and $Y(p)Z(p)$have the same eigenvalues. Here $v(p)$ and $i(p)$ constitute eigenvectors of $Z(p)Y(p)$ and $Y(p)Z(p)$ and which correspond to γ^2.

For any frequency p and any eigenvalue of the infinite dimension al product matrix $Z(p)Y(p)$ which corresponds to exact solution (γ^2_{real}) of the physical system, $Z(p)$ and $Y(p)$ with finite dimensions whose product has at least one eigenvalue corresponding to γ^2_{real} always exist [3]. Presented algebraic function approximation is based on equations of transmission line equivalence by considering this exactness of the MoM.

On the other hand the characteristic equation for finite dimensional (mxm) matrix $Z(p)Y(p)$, whose eigenvalue provides approximately the square of the propagation constant, transforms into an algebraic equation whose coefficients are polynomials depending on p with degree m. This definition can be illustrated as below.

$$\det\left[\gamma^2 I - Z(p)Y(p)\right] = g\left(\gamma^2, p\right) \quad (2)$$

where, det and I show determinant and unit matrix, respectively. If this polynomial in terms of γ^2 is equalized to zero, the characteristic equation is obtained. Then if this equation is multiplied by common denominator of the terms of the left side, a new form of the characteristic equation in the terms of powers of γ^2 where coefficients are polynomials of p is obtained [6-10]. If the left side of last equation is called $G(\gamma^2,p)$ and coefficient of i th power of γ^2 in the equation is called $a_{m-i}(p)$, the below equation is obtained.

$$G\left(\gamma^2, p\right) = a_0(p)\gamma^{2m} + a_1(p)\gamma^{2m-2} + \cdots + a_m(p) = 0 \quad (3)$$

The coefficients of these equations cannot be expressed in explicit form and they are calculated from determinant of the (mxm) matrix in (2) for any frequency. Approximate squares of the propagation constants are solutions of the characteristic equation. Thus, singular points of the solution are obtained simply from poles of $Z(p)Y(p)$ and zeros of discriminant of $G(\gamma^2,p)=0$. Because, according to algebraic function theory, singular points of roots of $G(\gamma^2,p)=0$ are either zeros of $a_0(p)$, which have to be poles of $Z(p)Y(p)$ due to eq. (2), or zeros of discriminant of eq. (3) [21]. In particular zeroes of $a_0(p)$ are poles or pole branch points and zeroes of the said discriminant are branch points.

Suppose we investigate the singularity at frequencies ω, in the neighborhood of ω_0, and it is seen that the phase constant $\beta(\omega)$ – assuming that the propagation constant is pure imaginary- diverges to infinity, while the working frequency ω approaches ω_0. An infinite value of $\beta(\omega)$ at ω_0 shows that ω_0 is a singular point and additionally $a_0(j\omega_0)$ at eq. (3) equals to zero.

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Additionally, as noted above $a_0(j\omega_0)$ equals to zero when frequency point $j\omega_0$ is pole or pole branch. But, Yener showed in [6] that $j\omega_0$ cannot be a pole branch and has to be a pole when $a_0(j\omega_0)=0$ for closed, lossless waveguide structures which do not couple longitudinal field components with transverse field components and whose loading medium has hermitian $\tilde{\varepsilon}$ and $\tilde{\mu}$ matrixes whose entries are rational functions of p. These properties of the waveguide structure are satisfied by our problem. Thus the singularity considered cannot be a pole branch point but has to be a pole. Similarly if at the investigated frequency ω_0, $\beta(\omega)$ displays a finite value but an infinite derivative with respect to frequency, then it can be concluded that the singularity is a branch point while $j\omega_0$ sets the discriminant of (3) equal to zero. If the propagation constant is complex, a branch point may exist also if the propagation constant is bounded but the attenuation constant has infinite derivative. The focus of the present study is on branch points of dispersion characteristics. Various configurations of dispersion characteristics which have branch points are illustrated in Figure 2.

The algebraic equation system for a closed waveguide structure and eigenvalues and characteristic equation of the system and also some fundamental features of algebraic function theory supply some advantages in order to scrutinize the dispersion characteristic of the structure. Additionally this algebraic equation system can be used for theoretical analysis. For instance, existence of complex wave [11] or co-existence of backward wave and complex wave for different waveguide structures [12, 13] were shown theoretically by using features of algebraic equation systems of the structures. In another example, by using algebraic equation system of the structure, Yener gave necessary and sufficient conditions for existence of backward waves in a heterogeneous and anisotropic material loaded metallic closed waveguide [4]. Another advantage of the approximation in theoretical analysis is that the propagation constants can be expressed as functions in the form of infinite series and thus it is possible to explore the problem in depth more efficiently using these simplified expressions for the propagation constant.

In our study, the branch points which exist on the dispersion curves at different frequency points and dispersion characteristics in the neighborhood of these points have been investigated by expressing the propagation constants in the form of infinite series, by the aid of algebraic function theory.

In the study, dispersion characteristics in the neighborhood of branch points belonging to gyro-electric medium loaded cylindrical metallic waveguide have been investigated. Similar treatment has been reported for isotropic heterogeneous media filled closed waveguides [10]. Also a gyro-magnetic material loaded waveguide has been considered in these references. In the presented study we extend this scope to include gyro-electric medium filled waveguides. For this purpose, the plasma column loaded cylindrical metallic waveguide is tackled as numerical example in the study. The cross-section of the plasma loaded waveguide structure is given in Figure 1.

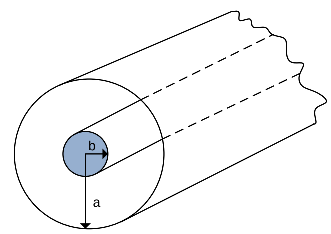

Figure 1. The cross-section of the plasma column loaded cylindrical waveguide

The plasma column loaded cylindrical metallic waveguide involve very complicated exact dispersion characteristics as given in [14, 15]. Nevertheless the algebraic function theory approach has been successfully implemented while observing all peculiarities of the dispersion curves of the structures regardless of the multitude of parameters that affected these curves for example normalized frequency, normalized cyclotron frequency, azimuthal variation, ratio of plasma column radius and waveguide radius. In paticular in the numerical example, a gyro-electric plasma column loaded cylindrical metallic waveguide has been used and its dispersion characteristics in the neighborhood of the branch points have been obtained from exact solution, transmission line equivalence method (or MoM) and algebraic function approximation (Puiseux series). Exact dispersion characteristics of the structure for guided wave modes and complex wave modes were presented by [14] and [15], respectively. Additionally, validation of the MoM at gyro-resonance frequency region and plasma resonance frequency regions for backward and forward guided wave modes of the structure were shown in the studies [16-19]. In [15], exact, MoM, quasitatic solutions belonging to the structure were given and validation of MoM results was achived by presenting comparatively complex dispersion curves obtained from these solutions.

Plasma waveguides can be also used as sensor in microwave devices and called as plasmonic sensor. The plasma cylindrical waveguide with three or more layers surrounded with gold has been investigated and the difference between the resonant wavelengths calculated by using finite element method and analytical method in [20]. These configurations have some advantageous as high amplitude sensitivity, a smaller propagation length and a better value of the detection limit of amplitude-based sensor.

In our study, briefly, dispersion curves in the neighborhood of branch points have been obtained also from algebraic function theory using Puiseux series whose coefficients are calculated by using equations system of the structure's transmission lines equivalent, which were presented and whose validation were shown in [15]. The Puiseux series enable to model the complex dispersion curves in the neighborhood of branch points with small number of terms. The Puiseux series is obtained from the characteristic equation of a coefficient matrix emerging in MoM. It differs from a curve-fitting say by the least squares approximation in that it has a mathematical basis whereas curve-fitting is a purely numeric technique. Thus it also enables a mathematical insight to the interpretation of MoM results beside representing the propagation constant numerically.

In section II, transition features between complex and non-complex wave modes and endpoint features of complex wave frequency interval are investigated. In section III, how the algebraic equations are used to obtain coefficients of Puiseux series is presented and validity of the method is shown on numerical examples. The study ends with the conclusion.

II. Examination Of Branch Points By Aid Of Algebraic Function Theory

In this section, dispersion characteristics in the neighborhood of the branch points are investigated by expressing the propagation constant in the form of infinite series and by aid of algebraic function theory. Additionally, some results for complex modes are extracted using algebraic function theory by investigating features of critical points where a frequency region with complex valued $\gamma(j\omega)$ ends. Transition from complex mode region to non-complex mode region where propagation constant is pure real or pure imaginary cannot occur at $j\omega_B$ point in the neighborhood of which $\gamma^2(p)$ is regular [6]. This property can be applied to any functional relations of $\gamma^2(p)$ (depending on p)which is either root of a algebraic equation or not.

Yener in [13] proved existence of a frequency region of complex modes adjacent to a frequency region of a backward wave mode by using algebraic function theory based on MoM when applied to a closed waveguide structure. In this section, some concepts given for complex wave modes and algebraic function theory by Yener are discussed and elaborated. In what follows, for two frequency points ω_B and $\omega_{B'}$, $\omega>\omega_B$ and $\omega>\omega_{B'}$ or $\omega<\omega_B$ and $\omega<\omega_{B'}$ indicate the immediate frequency neighborhoods of ω_B or $\omega_{B'}$.

A. Examination of Transition Properties Between Complex and Noncomplex Wave Mode Intervals

If $\gamma^2(p)$ is regular in the neighborhood of the point $j\omega_B$, a frequency region of a complex wave mode on frequency axis cannot end at ω_B point. Assume that ω_B is an endpoint of a complex mode interval. Then $\gamma^2(p)$ is regular in $|p-j\omega_B|<R$ neighborhood of $j\omega_B$, but $\gamma^2(j\omega)$ is real for one of the regions $\omega>\omega_B$ or $\omega<\omega_B$ while not for the other. For $0<\rho<R$, $\gamma^2(p)$ can be represented by a Taylor series in the region $|p-j\omega_B|<\rho$.

$$\gamma^2(p)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n\left(\gamma^2(j\omega_B)\right)}{dp^n}(p-j\omega_B)^n \tag{4}$$

For $p=j\omega$, the series transforms into equation (5)

$$\gamma^2(j\omega)=\gamma^2(j\omega_B)+\frac{d\gamma^2(j\omega_B)}{d\omega}(\omega-\omega_B)+\frac{1}{2}\frac{d^2\gamma^2(j\omega_B)}{d\omega^2}(\omega-\omega_B)^2+\cdots \tag{5}$$

Assume that $\gamma^2(j\omega)$ is real for $\omega>\omega_B$, This is possible if and only if $\left(\frac{d^n\left(\gamma^2(j\omega_B)\right)}{d\omega^n}\right)_{\omega=\omega_B}$ is real for n=1, 2, …, because of linear independence of $(\omega-\omega_B)^n$ multipliers. But, as seen in Equation (5), this situation precludes existence of complex value at $\omega<\omega_B$ since $\gamma^2(j\omega)$ is defined real for $\omega>\omega_B$. Hence transition from complex propagation constant to noncomplex, pure real or pure imaginary, cannot occur at $j\omega_B$ point in the neighborhood of which $\gamma^2(p)$ is regular. Besides $j\omega_B$ is a root of discriminant of $G(\gamma^2,p)=0$ since $\gamma^2(j\omega)$ is finite and multiple root (double root) of the algebraic equation at $j\omega_B$.

B. Examination of Some Properties for Endpoint of A Complex Wave Frequency Interval

In this section, ending conditions of complex wave modes regions modeled by Puiseux series expansion including no negative power term are investigated.

The series expansion with no negative power term for $\gamma^2(j\omega)$ in the vicinity of branch point $j\omega_B$ is given in Equation (6) [21].

$$\gamma^2(j\omega)=\gamma_1^2(j\omega_B)+\sum_{n=1}^{\infty} jA_n[\omega_B-\omega]^{n/q} \tag{6}$$

If $j\omega_B$ is an algebraic branch point where $\gamma^2(j\omega_B)$ is finite, $\gamma^2(j\omega_B)$ is a defective eigenvalue and this means that derivative of $\gamma^2(j\omega)$ at ω_B is infinite [6]. Moreover there exist always at least two real solutions on at least one side of $j\omega_B$ ($\omega>\omega_B$ or $\omega<\omega_B$) on the axis $j\omega$. In the structures under consideration there exists a multiple root of order 2 and hence the two real solutions approach mutual real eigenvalue $\gamma^2(j\omega_B)$ as double root (which means q=2) while ω approaches ω_B [6]. While β is imaginary part of the propagation constant and α is real part of the propagation constant, square of the propagation constant at ω_B where region of the complex propagation constants ends up transforms into Equation (7).

$$\gamma^2(j\omega_B)=-\beta^2(j\omega_B)\neq 0 \quad \text{or} \quad \gamma^2(j\omega_B)=\alpha^2(j\omega_B)\neq 0 \tag{7}$$

The dispersion characteristic in the vicinity of algebraic branch point $j\omega_B$ where $\gamma^2(p)$ is finite is given in Figure 2.

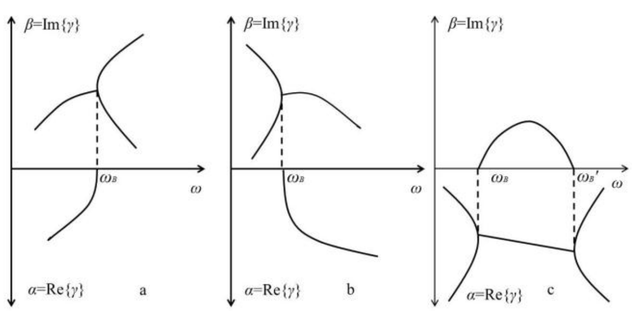

Figure 2. The dispersion characteristic in the vicinity of algebraic branch point $j\omega_B$ where $\gamma^2(j\omega_B)$ is finite.

The dispersion characteristic in the vicinity of algebraic branch point $j\omega_B$ where $\gamma^2(p)$ is finite is given Fig.1. The characteristic at Fig 1-a is an example for complex modes existing in isotropic dielectric column loaded cylindrical waveguide. The characteristic at Fig 1-b is an example for complex modes existing in plasma column loaded waveguide. Fig. 1-c contains the branch points of the same plasma column loaded cylindrical waveguide. In curve-c as a special case, the end points of complex modes frequency interval exist at splitting point of attenuation constant.

For branch point $j\omega_B$, and termination point of complex wave modes region ω_B,behavior of real and imaginary part of propagation constant can be investigated by considering equation (6) and taking q=2.

For the situation is given Fig 1-a, assume that coefficients A_n are pure real in order to ensure real $\gamma^2(j\omega)$ for $\omega>\omega_B$. Under this condition, complex wave modes exist on the region $\omega<\omega_B$. If real and imaginary parts of propagation constant are differentiated with respect to ω, three different cases are obtained ($\gamma^2(j\omega_B)<0$, $\gamma^2(j\omega_B)>0$ ve $\gamma^2(j\omega_B)=0$). For the case $\gamma^2(j\omega_B)=0$, both derivatives should be infinite. But it has been shown by Yener that infiniteness of both derivatives at the same time are impossible [6]. Fig. 1-a corresponds to the case $\gamma^2(j\omega_B)<0$. When $\gamma^2(j\omega_B)<0$, $\left|\lim_{\omega\to\omega_{B^-}}\left(\frac{d\alpha(j\omega)}{d\omega}\right)\right|=\infty$ and $\left|\lim_{\omega\to\omega_{B^-}}\left(\frac{d\beta(j\omega)}{d\omega}\right)\right|<\infty$ are obtained.

Similar procedure can be applied for the case given Fig 1-b. Assume that coefficients A_n are pure imaginary in order to ensure real $\gamma^2(j\omega)$ for $\omega<\omega_B$. In this case, complex wave modes exist on the region $\omega>\omega_B$. If real and imaginary parts of propagation constant are differentiated with respect to ω, $\left|\lim_{\omega\to\omega_{B^+}}\left(\frac{d\alpha(j\omega)}{d\omega}\right)\right|=\infty$ and, at the same time, $\left|\lim_{\omega\to\omega_{B^+}}\left(\frac{d\beta(j\omega)}{d\omega}\right)\right|<\infty$ for $\gamma^2(j\omega_B)<0$ are obtained since $\gamma^2(j\omega_B)<0$.

Fig 1-c shows special branch points of plasma column loaded cylindrical waveguide. In this curve, the termination points of complex mode frequency region exist at splitting points of attenuation constant. For this case, complex wave modes exist on the region $\omega_B<\omega<\omega_B'$. Let's consider the branch point at ω_B. Assume that coefficients A_n are pure imaginary in order to ensure real $\gamma^2(j\omega)$ for $\omega<\omega_B$. In this case, complex wave modes exist on the region $\omega>\omega_B$. If real and imaginary parts of propagation constant are differentiated with respect to ω, $\left|\lim_{\omega\to\omega_{B^+}}\left(\frac{d\alpha(j\omega)}{d\omega}\right)\right|<\infty$ and, at the same time, $\left|\lim_{\omega\to\omega_{B^+}}\left(\frac{d\beta(j\omega)}{d\omega}\right)\right|=\infty$ are obtained for $\gamma^2(j\omega_B)>0$ in Fig.1-c. Likewise, let's take the branch at ω_B' and substitute ω_B' instead of ω_B in equation (6). Assume that coefficients A_n are pure real in order to ensure real $\gamma^2(j\omega)$ for $\omega>\omega_{B'}$. In this case, complex wave modes exist on the region $\omega<\omega_{B'}$. If real and imaginary parts of propagation constant are differentiated with respect to ω, $\left|\lim_{\omega\to\omega_{B'^+}}\left(\frac{d\alpha(j\omega)}{d\omega}\right)\right|<\infty$ and, at the same time, $\left|\lim_{\omega\to\omega_{B'^+}}\left(\frac{d\beta(j\omega)}{d\omega}\right)\right|=\infty$ are obtained for $\gamma^2(j\omega_{B'})>0$ in Fig.1-c.

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C. Obtaining The Coefficients of Puiseux Series

It has been presented in previous section that the partial differential equations system for gyrotropic medium loaded closed waveguide transforms into linear algebraic equations system given in (1). It has also been presented that the characteristic equation for (mxm) dimension matrix of $Z(p)Y(p)$ in equation (1) whose eigenvalues correspond approximately to the square of the propagation constant transforms into an algebraic equation in γ^2 with the degree m whose coefficients are polynomials depending on p. The characteristic equation of this system is given equation (2). Emergence of an algebraic equation as a consequence of transmission line equivalence method (or MoM) is the basis of applied method in the study. Solutions of the algebraic equation are approximate values of the squares of the propagation constants. Hence the singular points of the solution are obtained simply from poles of $Z(p)Y(p)$ and zeros of discriminant of $G(\gamma^2, p)=0$ derived from the characteristic equation for $Z(p)Y(p)$, by virtue of the algebraic function theory.

The Puiseux series enables to model the complex dispersion curves in the neighborhood of branch points with a small number of coefficients. The coefficients of Puiseux series are obtained using the transmission line equations system belonging to the structure and the characteristic equation of this system. In the following we shall compute only the coefficients of the first two terms of the Puiseux series. We shall see in section III that this will be sufficient to represent the propagation constant accurately in the neighborhood of the branch point. The Puiseux series expansion of propagation constant in the vicinity of a branch point is given below.

$$\gamma^2(p)=\gamma_1^2(j\omega_0)+A_1\sqrt{p-j\omega_0}+A_2(p-j\omega_0)+\cdots+A_n(p-j\omega_0)^{n/2}+\cdots \tag{8}$$

Here, ω_0 and $A_1\ldots A_n$ are frequency of branch point and coefficients of the series expansion, respectively. If Equation (8) is rewritten by taking $p'=\sqrt{p-j\omega_0}$ in order to obtain the coefficients of the Puiseux series, new form of the equation is obtained as below.

$$\gamma^2(p)=\gamma_1^2(j\omega_0)+A_1p'+A_2(p')^2+\cdots+A_n(p')^n+\cdots \tag{9}$$

In order to obtain coefficient A_1, let's differentiate both sides of Equation (9) with respect to p' at p'=0.

$$A_1=\left[\frac{d\gamma^2(p)}{dp'}\right]_{p'=0}=\left[\frac{d\gamma^2(p)}{dp}(2p')\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}} \tag{10}$$

Let's take the characteristic equation of the system given in Equation (3) to solve the equation above. Equation (3) is an implicit function depending on γ^2 and p.

So Equation (10) can be written as below by using derivative of implicit function.

$$A_1=\left[-\frac{G_p}{G_{\gamma^2}}(2p')\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}} \tag{11}$$

It is known that roots of the implicit function ($G(\gamma^2,p)=det[Z(p)Y(p)-\gamma^2(p)I]=0$) correspond to the propagation constants. In the expression, G_p and G_{γ^2} are derivative of $G(\gamma^2,p)$ with respect to p and γ^2. It is beneficial to remind that $G(\gamma^2,p)$ has two form given in Equation (3) and $G(\gamma^2,p)=det[Z(p)Y(p)-\gamma^2(p)I]=0$. Therefore the derivatives can be obtained by using any one of both. Then expression in Equation (3) for the matrix of $Z(p)Y(p)$ with (mxm) dimension can be arranged in terms of roots of the characteristic equation.

$$G(\gamma^2,p)=(\gamma^2(p)-\gamma_1^2(p))(\gamma^2(p)-\gamma_2^2(p))(\gamma^2(p)-\gamma_3^2(p))\cdots(\gamma^2(p)-\gamma_m^2(p)) \tag{12}$$

The implicit function can be expressed as below in the vicinity of $p=j\omega_0$ and $\gamma^2=\gamma_1^2(j\omega_0)$, because there is multiple root ($\gamma_1^2(j\omega_0)=\gamma_2^2(j\omega_0)$) with second degree at branch point, $p=j\omega_0$.

$$G(\gamma^2,p)=(\gamma^2(p)-\gamma_1^2(j\omega_0))^2(\gamma^2(p)-\gamma_3^2(j\omega_0))\cdots(\gamma^2(p)-\gamma_m^2(j\omega_0)) \tag{13}$$

At $\gamma^2=\gamma_1^2$, $p=j\omega_0$, derivative of G with respect to γ^2 (G_{γ^2}) equals to zero owing to $\gamma^2=\gamma_1^2(j\omega_0)$ at $p=j\omega_0$. G_{γ^2} in the vicinity of multiple root point $p=j\omega_0$ and $\gamma^2=\gamma_1^2(j\omega_0)$ equals approximately the expression below.

$$\begin{aligned}\frac{\partial G(\gamma^2,p)}{\partial\gamma^2}&=2(\gamma^2(p)-\gamma_1^2(j\omega_0))(\gamma_1^2(j\omega_0)-\gamma_3^2(j\omega_0))\cdots(\gamma_1^2(j\omega_0)-\gamma_m^2(j\omega_0))\\&+(\gamma^2(p)-\gamma_1^2(j\omega_0))^2(\gamma_1^2(j\omega_0)-\gamma_4^2(j\omega_0))\cdots(\gamma_1^2(j\omega_0)-\gamma_m^2(j\omega_0))+\cdots\\&+(\gamma^2(p)-\gamma_1^2(j\omega_0))^2(\gamma_1^2(j\omega_0)-\gamma_3^2(j\omega_0))\cdots(\gamma_1^2(j\omega_0)-\gamma_{m-1}^2(j\omega_0))\end{aligned} \tag{14}$$

Because common expression $(\gamma^2(p)-\gamma_1^2(j\omega_0))^2$ in second and later summations of the equation has very small value in the vicinity of $p=j\omega_0$, G_{γ^2} is obtained approximately as below,

$$\frac{\partial G(\gamma^2,p)}{\partial\gamma^2}=2(\gamma^2(p)-\gamma_1^2(j\omega_0))(\gamma_1^2(j\omega_0)-\gamma_3^2(j\omega_0))\cdots(\gamma_1^2(j\omega_0)-\gamma_m^2(j\omega_0)) \tag{15}$$

and if E is determined as below,

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$$E=\left(\gamma_1^2(j\omega_0)-\gamma_3^2(j\omega_0)\right)\left(\gamma_1^2(j\omega_0)-\gamma_4^2(j\omega_0)\right)\cdots\left(\gamma_1^2(j\omega_0)-\gamma_m^2(j\omega_0)\right) \quad (16)$$

Equation (15) transforms to Equation (17).

$$G_{\gamma^2}=2\left(\gamma^2(p)-\gamma_1^2(j\omega_0)\right)E \quad (17)$$

If Equation (17) is substituted in Equation (11), the expression below is obtained.

$$A_1=\left[-\frac{G_p}{2\left(\gamma^2(p)-\gamma_1^2(j\omega_0)\right)E}(2p')\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}} \quad (18)$$

Besides, if the expression, $\gamma^2(p)-\gamma_1^2(j\omega_0)$, is extracted from Equation (9) in the vicinity of $p=j\omega_0$ and substituted in Equation (18) and final expression is arranged for $p'=0$ ($p=j\omega_0$), Equation (19) is obtained.

$$A_1=\left[-\frac{G_p}{2\left(A_1p'+A_2(p')^2+\cdots+A_n(p')^n\right)E}(2p')\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}} \quad (19)$$

If the expression is arranged again for A_1, Equation (20) is obtained.

$$A_1=\left[-\frac{G_p}{A_1E}\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}}=\sqrt{-\frac{G_p}{E}} \quad (20)$$

Derivative of G with respect to p in the equation can be obtained from derivative of determinant by using definition of $G(\gamma^2,p)=det[Z(p)Y(p)-\gamma^2(p)I]$. As known derivative of determinant for a matrix A is equal to $d\det(A)=\mathrm{tr}(\mathrm{Adj}(A)dA)$. Here, dA, tr and Adj stands the differential of A, trace of matrix and adjugate of matrix, respectively Then G_p is given below.

$$G_p=tr\left(Adj\left(Z(p)Y(p)-\gamma^2(p)I\right)\left(\frac{dZ(p)}{dp}Y(p)+Z(p)\frac{dY(p)}{dp}-\frac{d\gamma^2(p)}{dp}I-\gamma^2(p)\frac{dI}{dp}\right)\right) \quad (21)$$

Derivative of unit matrix with respect to p in Equation (21) equals to zero. Because of $\partial\gamma^2/\partial p=0$, explicitly derivative of G (derivative of determinant) with respect to γ^2 can be written as below.

$$tr\left(Adj\left(Z(p)Y(p)-\gamma^2(p)I\right)\left(\frac{\partial(Z(p)Y(p))}{\partial\gamma^2}-I\frac{\partial\gamma^2}{\partial\gamma^2}\right)\right)\frac{\partial\gamma^2}{\partial p}=0 \quad (22)$$

Derivative of product $Z(p)$ $Y(p)$ with respect to γ^2 equals to zero because $Z(p)$ and $Y(p)$ are independent from γ^2.

Additionally derivative of γ^2 with respect to itself equals to unit so Equation (22) transforms into Equation (23).

$$tr\left(Adj\left(Z(p)Y(p)-\gamma^2(p)I\right)\right)\frac{\partial\gamma^2}{\partial p}=0 \quad (23)$$

where, due to $tr\left(Adj\left(Z(p)Y(p)-\gamma^2(p)I\right)\right)\neq 0$, in order to satisfy Equation (23), $\partial\gamma^2/\partial p=0$ has to hold. Consequently G_p transforms into Equation (24).

$$G_p=tr\left(Adj\left(Z(p)Y(p)-\gamma^2(p)I\right)\left(\frac{dZ(p)}{dp}Y(p)+Z(p)\frac{dY(p)}{dp}\right)\right) \quad (24)$$

A_1 can be obtained by using Equations (16) and (24). If we use Equation (24) and definition of trace of a matrix which is sum of elements on the main diagonal, the coefficient A_1 is obtained as Equation (25), below.

$$A_1=\sqrt{-tr\left(\frac{Adj\left(Z(p)Y(p)-\gamma^2(p)I\right)}{E}\left(\frac{dZ(p)}{dp}Y(p)+Z(p)\frac{dY(p)}{dp}\right)\right)} \quad (25)$$

In order to obtain coefficient A_2, if we take second order derivative of both side of Equation (9) with respect to p' at $p'=0$, we obtain Equation (26).

$$A_2=\frac{1}{2}\left[\frac{d^2\gamma^2(p')}{dp'^2}\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}}=\frac{1}{2}\frac{d}{dp'}\left[\frac{d\gamma^2(p)}{dp}(2p')\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}}=\frac{1}{2}\left[\frac{d^2\gamma^2(p)}{dp^2}(2p')^2+2\frac{d\gamma^2(p)}{dp}\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}} \quad (26)$$

Let's write second order derivative of implicit function [22],

$$\frac{d^2\gamma^2(p)}{dp^2}=-\frac{G_{pp}\left(G_{\gamma^2}\right)^2-2G_{\gamma^2p}G_pG_{\gamma^2}+G_{\gamma^2\gamma^2}\left(G_p\right)^2}{\left(G_{\gamma^2}\right)^3} \quad (27)$$

and substitute it with first order derivative of implicit function in Equation (26), then,

$$A_2=\frac{1}{2}\left[-\frac{G_{pp}}{G_{\gamma^2}}(2p')^2+\frac{2G_{\gamma^2p}}{G_{\gamma^2}}\frac{G_p}{G_{\gamma^2}}(2p')^2-\frac{G_{\gamma^2\gamma^2}}{G_{\gamma^2}}\left(\frac{G_p}{G_{\gamma^2}}\right)^2(2p')^2-2\frac{G_p}{G_{\gamma^2}}\right]_{\substack{p=j\omega_0\\ \gamma^2=\gamma_1^2(j\omega_0)}} \quad (28)$$

Second order derivative of G with respect to γ^2 can be written from Equation (17) as below.

$$G_{\gamma^2\gamma^2}=2E \quad (29)$$

If equation (28) is rearranged using Equations (11), (17) and (29), A_2 is obtained as Equation (30).

$$A_2 = \frac{1}{4\left(\gamma^2(p) - \gamma_1^2(j\omega_0)\right)E}\left[-4G_{pp}\left(p'\right)^2 + 4G_{\gamma^2 p}\left(-A_1\right)p' - 2EA_1^2 - 2G_p\right]_{\substack{p=j\omega_0 \\ \gamma^2=\gamma_1^2(j\omega_0)}} \tag{30}$$

Using Equations (9) and (20), expression A_2 transforms into below equation.

$$A_2 = \frac{1}{E\left(A_1 + A_2 p' + A_3\left(p'\right)^2 + \cdots\right)}\left[-G_{pp}p' - G_{\gamma^2 p}A_1\right]_{\substack{p=j\omega_0 \\ \gamma^2=\gamma_1^2(j\omega_0)}} \tag{31}$$

A_2 for p'=0 equals to below expression.

$$A_2 = -\frac{G_{\gamma^2 p}}{E} \tag{32}$$

If we use derivative of G with respect to γ^2, utilized in equations (22) and (23), and differentiate it with respect to p, $G_{\gamma^2 p}$ is obtained as below.

$$\frac{\partial^2 G(\gamma^2, p)}{\partial\gamma^2\partial p} = -\frac{\partial\left(tr\left(Adj\left(Z(p)Y(p) - \gamma^2(p)I\right)\right)\right)}{\partial p} \tag{33}$$

If D is a matrix and $tr(D)$ means trace of the matrix, it is determined that $\partial tr(D) = tr(\partial D)$. Then $G_{\gamma^2 p}$ can be written as below.

$$\frac{\partial^2 G(\gamma^2, p)}{\partial\gamma^2\partial p} = -tr\left(\frac{\partial Adj\left(Z(p)Y(p) - \gamma^2(p)I\right)}{\partial p}\right) \tag{34}$$

The expression $Z(p)Y(p)-\gamma^2(p)I$ in Equation (34) is a singular matrix, due to $G(\gamma^2,p)=det(Z(p)Y(p)-\gamma^2(p)I)=0$. Definition of derivative of the adjugate matrix contains the inverse matrix. Therefore the classical formulation of derivative of the adjugate matrix cannot be used for Equation (34). Nevertheless derivative of adjugate of a singular/nonsingular matrix can be obtained from definition of adjugate matrix which is the transpose of its cofactor matrix [23]. Suppose that B is a nonsingular or singular matrix with nxn dimension and C is the cofactor matrix of B and i, j are row and column indices, respectively.

$$\mathrm{Adj}(B) = C^{\mathrm{T}} \text{ and } C_{\mathrm{ij}} = (-1)^{\mathrm{i+j}}\det\left(b_{\mathrm{ij}}\right) \tag{35}$$

here, b_{ij} is the submatrix of B obtained by deleting i.th row and j.th column and which we shall henceforth call (i,j) submatrix of any matrix and denote by $[\ ]_{ij}$ instead of lower case with subscript $_{ij}$. Namely, $b_{ij}=[B]_{ij}$. Let's suppose that D is derivative of adjugate of a nonsingular or singular matrix B, $D = dAdj(B)/dp$. When definitions of adjugate, cofactor, determinant and submatrix of a matrix are substituted in matrix D, elements of matrix D can be written as below.

$$D_{ij} = (-1)^{i+j}\frac{d\det\left([B]_{ji}\right)}{dp} \tag{36}$$

From derivative of determinant, Equation (36) transforms into Equation (37).

$$D_{ij} = (-1)^{i+j}tr\left(Adj\left([B]_{ji}\right)\frac{d[B]_{ji}}{dp}\right) \tag{37}$$

(i,j) element of matrix D in Equation (37) contains derivative of submatrix of B. It is inconvenient to find derivative of (i,j) submatrix at every turn. Nevertheless it can be proved that derivative of (i,j) submatrix of the main matrix equals to (i,j) submatrix of derivative of the main matrix, $\frac{d[B]_{ij}}{dp} = \left[\frac{dB}{dp}\right]_{ij}$ [23]. Consequently elements of D which is derivative of adjugate of a nonsingular or singular matrix B is obtained as below.

$$D_{ij} = (-1)^{i+j}tr\left(Adj\left([B]_{ji}\right)\left[\frac{dB}{dp}\right]_{ji}\right) \tag{38}$$

By using the result in Equation (38) and definition of trace of a matrix, second order derivative of implicit function $G(\gamma^2,p)$ with respect to γ^2 and p given in Equation (34)is obtained as Equation (39).

$$G_{\gamma^2 p} = -\sum_{i=1}^{n} tr\left(Adj\left(\left[Z(p)Y(p) - \gamma^2(p)I\right]_{ii}\right)\left[\frac{dZ(p)}{dp}Y(p) + Z(p)\frac{dY(p)}{dp}\right]_{ii}\right) \tag{39}$$

So coefficient A_2 is obtained as Equation (40).

$$A_2 = -\frac{G_{\gamma^2 p}}{E} = \sum_{i=1}^{n} tr\left(\frac{Adj\left(\left[Z(p)Y(p) - \gamma^2(p)I\right]_{ii}\right)}{E}\left[\frac{dZ(p)}{dp}Y(p) + Z(p)\frac{dY(p)}{dp}\right]_{ii}\right) \tag{40}$$

The denominator E in Equations (25) and (40) is defined as the product of differences of the roots from the double root as given by Equation (16). Number of factors in Equation (16) equals (m-2) when product matrix $Z(p)Y(p)$ has dimension (mxm). While m gets larger, E approaches infinity due to limit of the largest number in numerical calculations. For example, the range for double in MATLAB is -1.79769e+308 to -2.22507e-308 and 2.22507e-308 to 1.79769e+308. Similarly, elements of adjugate matrices in Equations (25) and (40) approach infinity in numerical computations, while dimension of product matrix of $Z(p)Y(p)$ gets larger. Large dimension of the matrix, determined by number of TE and TM modes (or eigen functions) of empty guide, causes numerical computation problems.

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In order to elaborate on this problem, suppose that (mxm) dimensional matrix inside of the adjugate in the equations is B and its cofactor matrix is C and the (i,j) submatrix of B is $[B]_{ij}$. Then elements of the matrix $Adj()/E$ in the equations can be written as below.

$$\frac{C_{ij}}{E}=(-1)^{i+j}\frac{\det([B]_{ij})}{E} \tag{41}$$

Where, the ratio det()/E approaches the indeterminate form ∞/∞ in numerical computations, while dimension of the algebraic equations system gets larger because of the above explained reasons. Actually the value of this ratio is significant. In order to avoid the indeterminate form ∞/∞ in numerical computations, submatrix matrix (b_{ij}) with $(m\text{-}1)\text{x}(m\text{-}1)$ dimension can be factorized by using Singular Value Decomposition (SVD).

$C_{ij} = (-1)^{i+j}\det(b_{ij}) = (-1)^{i+j}\det(U\Sigma V^*)$, where U is an $(m\text{-}1)\text{x}(m\text{-}1)$ real or complex unitary matrix, V^* (the conjugate transpose of V) is an $(m\text{-}1)\text{x}(m\text{-}1)$ real or complex unitary matrix and Σ is an $(m\text{-}1)\text{x}(m\text{-}1)$ diagonal matrix with non-negative real numbers on the diagonal. Then Equation (41) can be written as below [24].

$$\frac{C_{ij}}{E}=(-1)^{i+j}\det(U)\frac{\det(\Sigma)}{E}\det(V^*) \tag{42}$$

Since the diagonal matrix Σ is defined as $\Sigma=\text{diag}(\sigma_1, \sigma_2,\dots, \sigma_{m\text{-}1})$ and its determinant is the product of diagonal elements and the denominator E is the products of differences of the roots as in Equation (16), Equation (42) can be written as Equation (43).

$$\frac{C_{ij}}{E}=(-1)^{i+j}\det(U)\left[\frac{\sigma_1*\sigma_2}{\gamma_1^2(j\omega_0)-\gamma_m^2(j\omega_0)}\prod_{i=3}^{m-1}\frac{\sigma_i}{\gamma_1^2(j\omega_0)-\gamma_i^2(j\omega_0)}\right]\det(V^*) \tag{43}$$

Where, Π is the product operator. Consequently it is possible to avoid undetermined value of $Adj()/E$ by using Equation (43) in numerical computation of expressions A_1 and A_2 because each ratio contributes a determinate factor to the overall product.

III. Dispersion Curves Of Plasma Column Loaded Cylindrical Waveguide In The Neighborhood Of Branch Points Using Puiseux Series

It is presented above that the dispersion characteristic in the neighborhood of branch points which exist for closed waveguide structures at different frequencies can be modeled by Puiseux series with a small number of coefficients obtained from the transmission line equations system belonging to the structure and the characteristic equation of this system.

In this section, the method is applied to plasma column loaded cylindrical metallic waveguide. The cross-section of the structure is given in Figure 1. The permittivity in the waveguide is given in equation (44).

$$\varepsilon(\mathrm{r})=\begin{cases}\tilde{\varepsilon} & 0<\mathrm{r}<\mathrm{b}\\ \varepsilon_0 & \mathrm{b}<\mathrm{r}<\mathrm{a}\end{cases} \tag{44}$$

where, the plasma column tensor permittivity is given in equation (45).

$$\tilde{\varepsilon}=\varepsilon_0\begin{bmatrix}\varepsilon_1 & j\varepsilon_2 & 0\\ -j\varepsilon_2 & \varepsilon_1 & 0\\ 0 & 0 & \varepsilon_3\end{bmatrix} \tag{45}$$

here,

$$\varepsilon_1=1+\frac{1}{\mathrm{R}^2-\Omega^2},\qquad \varepsilon_2=\frac{-\mathrm{R}}{\Omega(\mathrm{R}^2-\Omega^2)},\qquad \varepsilon_3=1-\frac{1}{\Omega^2} \tag{46}$$

and, Ω is the normalized frequency and R is the normalized cyclotron frequency. Expressions of Ω and R are given in (47).

$$\Omega=\frac{\omega}{\omega p},\qquad \mathrm{R}=\frac{\omega c}{\omega p} \tag{47}$$

Where, ω_p is the plasma frequency and ω_c is the cyclotron frequency. Permeability of the structure for any point of the cross-section is equal to free space permeability (μ_0). The exact solution and transmission line equations and the complex dispersion curves of the structure on normalized frequency-normalized propagation constant plane for different parameters were given in [15]. In that study the frequency points where the dispersion curves bifurcate and the complex wave modes appear were reported in detail. In the present study, the dispersion curves in the neighborhood of branch points for plasma column loaded cylindrical waveguide are obtained from the exact solution, the MoM and the Puiseux series and they are presented in the same figure.

As mentioned above, dimension of transmission line equations system given in Equation (1) is determined with number of TE and TM modes (or eigen functions) of empty guide. As is known, while number of eigenfunctions gets larger, duration of calculation gets larger, nevertheless compliance of the method to the exact solution gets better. In Figure 2, dispersion curves in the neighborhood of branch point for different number of eigenfunctions are given to show validity of presented method and effect of number of eigenfunction.

First numeric example of the presented method is given for plasma column loaded cylindrical waveguide with the parameter R=0,5 (relatively weak magnetic field) and s_0=0,1.

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Complex dispersion curve for the structure exists in the frequency interval $0,988762<\Omega<,015956$ [15]. In the expressions, s_0 is the plasma ratio in the guide ($s_0=b/a$) and b, a are the plasma radius, the waveguide radius, respectively. In all numerical computations, the waveguide radius a=3cm, the plasma frequency $\omega_p=10^{10}$ rad/s have been taken. Dispersion curves of plasma column loaded cylindrical waveguide obtained from the exact solution, the transmission line equivalent for m=150 and m=1000 (MoM_{150} and MoM_{1000}) and the Puiseux series ($Puiseux_{150}$ and $Puiseux_{1000}$) in the neighborhood of the branch point at Ω_0=0,988762 are given in Figure 3.

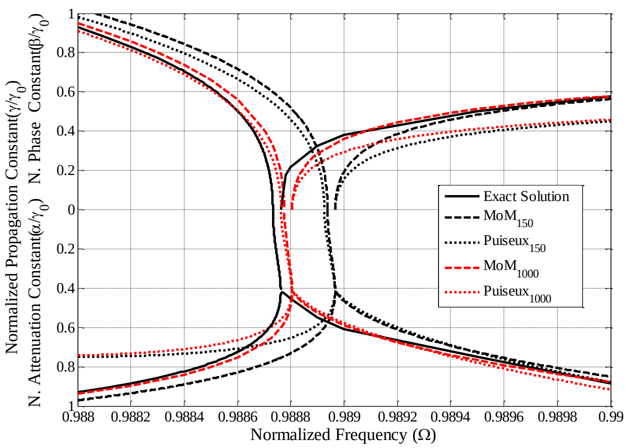

Figure 3. Dispersion curves obtained from the exact solution, the MoM and the Puiseux series in the neighborhood of branch point at Ω_0=0,988762 for R=0,5 and s_0=0,1

The exact branch point for plasma column loaded cylindrical waveguide with R=0,5 and s_0=0,1 appears at Ω_0=0,988762. Dispersion curve obtained from MoM for m=150 bifurcates at Ω_0=0,98896. The coefficients of Puiseux series obtained by using Equations (25) and (40) for m=150 are A_1=0,19337(1+i) ve A_2=-4,2981x10^{-5}i. The bifurcation for m=1000 occurs at Ω_0=0,988805. The coefficients of Puiseux series for m=1000 are A_1=0,191004(1+i) ve A_2=-4,2808x10^{-5}i. As is seen in the figure, the results obtained from MoM with larger number of eigenfunctions agree better with the exact solution than the results obtained from MoM with smaller number of eigenfunctions. The result obtained from Puiseux series with larger number of eigen functions also agrees better with the exact solution, because the calculating coefficients of Puiseux series is based on series transmission line equivalence equations.

Another example is given for plasma column loaded cylindrical waveguide with the parameter R=1,5 (relatively strong magnetic field) and s_0=0,5.

The dispersion curves obtained from the exact solution, the MoM and the Puiseux series for 150 TE and 150 TM modes of empty waveguide (m=150) are presented in Figure 4. The branch point of exact dispersion occurs at Ω_0=1,59244. The branch point for the MoM occurs at Ω_0=1,59309. The coefficients of Puiseux series obtained using Equations (25) and (40) equal to A_1=0,70297 *(1+i) and A_2=-4,34551x10^{-5}i.

As is seen in the figures, dispersion curves obtained from the MoM and the Puiseux series agree fairly well with the exact solution in the neighbourhood of branch point. The results obtained from the Puiseux series agree less with the MoM and also the exact solution, while moving away from the branch point. It is shown with the examples that dispersion curves in the neighborhood of branch point can be modeled by Puiseux series with only two coefficients.

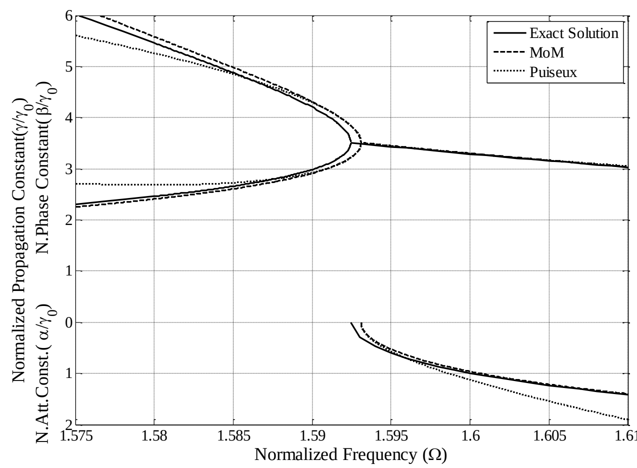

Figure 4. Dispersion curves obtained from the exact solution, the MoM and the Puiseux series in the neighborhood of branch point at Ω_0=1,59244 for R=1,5 and s_0=0,5

IV. CONCLUSION

In the study, dispersion characteristics in the neighbourhood of branch points for gyro-electric medium loaded cylindrical metallic waveguides are investigated using Puiseux series with a small number of coefficients obtained from the structure's system of transmission line equivalence equations and the characteristic equation belonging to this system. Some features of complex wave modes are investigated by using the algebraic equation which emerges as a consequence of transmission line equivalence method (or MoM). These features include transition properties between complex and noncomplex modes and endpoint features of complex wave frequency interval.

Next, it is presented that the dispersion characteristics of the structure in the neighbourhood of branch points can be obtained from expressing the propagation constants in the form of infinite series by the aid of algebraic function theory. In this approximation, the coefficients of Puiseux series are obtained using the equations system corresponding to the transmission line equivalence belonging to the structure and the characteristic equation of this system. Plasma column loaded cylindrical metallic waveguide whose filling is of the gyro-electric media class is used for numerical examples. Validation of the presented method is realized by giving comparatively the obtained results from exact solution, MoM and Puiseux series. Additionally, the effect of larger number of eigenfunctions on compliance of results obtained from MoM and Puiseux series with the exact solution is shown.

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