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New Trends on Injective Tensor Products of Separeted Convex Spaces.

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***Abstract*-- This paper presents the study of Injective tensor products of separated convex spaces. By defining, the compatible Topology , Inductive Topology and €- Topology with complete €- Tensor Product, it is proved in this paper that an Injective Tensor Product of two separated convex spaces is isomorphic to a sub space of an Injective Tensor Product of Products of two suitable families of Banach spaces.**

***Keywords*--Injective Tensor Product, Semi-norm, Continuous bilinear form, Isomorphism, Banach space.**

I. INTRODUCTION

Halub (1) and Kothe (2,3) are the pioneer workers of the present area . In fact , the present work is the extension of work done by Schechter, M.(4), Srivastava et al.(5), Srivastava et al.(6) , Srivastava et al.(7) and Srivastava et al.(8) . In this paper, we have studied new trends on injective tensor products of separated convex spaces.

Here, we use the following definitions and fundamental ideas:

Definition 1 (Grothendieck) : A locally convex topology T on $E \otimes F$ is said to be compatible with the tensor product $E \otimes F$ if and only if

(i) The canonical map $\psi : E \times F \to E \underset{T}{\otimes} F$ is separately continuous,

(ii) $u \otimes v$ with $u \in E'$ and $v \in F'$ is continuous on $E \underset{T}{\otimes} F$,

(iii) All the sets $G_1 \otimes G_2$ are equicontinuous on $E \underset{T}{\otimes} F$ for G_1, G_2 equicontinuous subsets of E', F' respectively.

FACT 1: Condition (i) is equivalent to the requirement

$$(E \underset{T}{\otimes} F)' \subseteq \mathcal{L}(E \times F).$$

Proof: Assume condition (i). Let x_0, W be given. Then there exists V with $\psi(x_0, V) = x_0 \otimes V \subseteq W$. So if

$$\hat{B} \in (E \underset{T}{\otimes} F)' \quad \text{with} \quad |\hat{B}(W)| \leq 1 ,$$

Then $|B(x_0, V)| \leq 1$ and B is separately continuous.

Assume $(E \underset{T}{\otimes} F)' \subseteq \mathcal{L}(E \times F)$. Choose $\hat{B} \in (E \underset{T}{\otimes} F)'$ and let W be such that $|\hat{B}(W)| \leq 1$. Given x_0, the assumption implies there exists V so that $|B(x_0, V)| \leq 1$ and thus also $|\hat{B}(x_0 \otimes V)| \leq 1$. So $x_0 \otimes V \subseteq W$. Hence $\psi(x_0, V) = x_0 \otimes V \subseteq W$ and ψ is separately continuous.

FACT 2 : Condition (ii) is equivalent to the requirement $E' \otimes F' \subseteq (E \underset{T}{\otimes} F)'$.

Remark 1 : We have seen, therefore, that conditions (i) and (ii) of the definition 1 are equivalent to the requirement

$$E' \otimes F' \subseteq (E \underset{T}{\otimes} F)' \subseteq £(E \times F).$$

Proposition 1: A locally convex topology T on $E \otimes F$ is compatible if and only if T is a topology of uniform convergence on a class m of subsets M of a space $H \subseteq £(E \times F)$ satisfying the following two conditions :

(a) For every $x_0 \in E$, $\tilde{M}(x_0)$ is equicontinuous in F'. For every $y_0 \in F$, $\tilde{\tilde{M}}(y_0)$ is equicontinuous in E'.

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(b) m contains the class m_0 of all sets $G_1 \otimes G_2$.

Proof: Suppose T is compatible. Using the remark 1 yields $(E \otimes_T F)' \subseteq \mathcal{L}(E \times F)$. T is the topology of uniform convergence on equicontinuous subsets of $(E \otimes_T F)' \subseteq \mathcal{L}(E \times F)$.

Condition (b) is satisfied since the sets $G_1 \otimes G_2$ are equicontinuous by condition (iii) of definition 1. To see condition (a) is satisfied, we use (i) of definition 1. Let x_0 be given. Now M equicontinuous on $E \otimes_T F$ implies there exists a W with $|\hat{M}(W)| \leq 1$. By (i) of definition 1, choose V, ρ with $\rho x_0 \otimes V \subseteq W$. Then $|\hat{M}(\rho x_0 \otimes V)| \leq 1$ or equivalently

$|\hat{M}(\rho x_0)(V)| \leq 1$ or $\tilde{M}(\rho x_0) \subseteq V$ so $\tilde{M}(x_0)$ is equicontinuous.

Conversely suppose the conditions given in this proposition hold. It follows immediately that the condition of the definition are met.

The role of condition (iii) in the definition 1, is it a "natural" condition .

Definition 2 : The finest compatible topology on $E \otimes F$ is the topology of uniform convergence on the class of all separaetly equicontinuous subsets of $\pounds(E \times F)$. This topology is called the inductive topology and is denoted by $E \otimes_i F$.

This is the finest topology such that the canonical map $\psi : E \times F \rightarrow E \otimes F$ is separately continuous. Also

$$(E \otimes_i F)' = \pounds(E \times F).$$

[Let m be the class of all subsets of $\pounds(E \times F)$ satisfying (a) and (b) of proposition 1].

We denote $E \tilde{\otimes}_i F$ = complete inductive tensor product.

Definition3: The weakest compatible topology on $E \otimes F$ is the topology of uniform convergence on the class of all sets $G_1 \otimes G_2$ in $E' \otimes F'$ where G_1 and G_2 are equicontinuous sets in E' and F' respectively. This topology is called the ϵ-topology and is denoted by $E \otimes_\epsilon F$.

[Let $m = m_0$ where m_0 is the class of all G_1 G_2 defined in (b) of proposition 1].

We denote $E \tilde{\otimes}_\epsilon F$ = complete ϵ-tensor product.

Remark 4 : The equality $E \otimes_i F = E \otimes_\pi F$ holds in the following cases :

(a) E and F complete metrizable
(b) E and F barreled metrizable.
(c) E and F barreled (DF)- spaces.

Proof: The three major continuity theorems namely Bourbaki continuity theorem, assume that under any of the conditions (a), (b) or (c) above, the equality $\pounds(E \times F) = \beta(E \times F)$ is valid, and the seprately equicontinuous and equicontinuous sets coincide. This is sufficient to conclude

$$E \otimes_i F = E \otimes_\pi F.$$

This proves the remark 4.

Here we note that the equality $E \otimes_i F = E \otimes_\pi F$ does not necessarily hold for E and F normed spaces.

Proposition 2: The ϵ-topology on $E \otimes F$ is given by the semi- norms

$$\epsilon_{G_1 \otimes G_2}(\Sigma x_i \otimes y_i) = \sup_{\substack{u \in G_1 \\ v \in G_2}} |\Sigma (u x_i)(v y_i)| ,$$

Where G_1, G_2 are equicontinuous sets in E', F'.

Proposition 3 : If E and F are normed, so is $E \otimes_\epsilon F$ and the

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ϵ - norm on $E \otimes F$ is given by

$$||\Sigma x_i \otimes y_i||_\epsilon = \sup_{\substack{||u||\le 1 \\ ||v||\le 1}} |\Sigma (u x_i)(v y_i)| .$$

In particular, $||x \otimes y||_\epsilon = ||x||.||y||$.

Remark 5 : The semi-norms and norms defined in proposition 2 and 3 are independent of the representation $\Sigma x_i \otimes y_i$ of an element in $E \otimes F$ since the value of a linear functional $u \otimes v$ at a point $z = \Sigma x_i \otimes y_i$ is independent of representation.

Proposition 4 : $E \otimes_\epsilon F$ is a linear subspace (both algebraically and topologically) of $£_e(E'_S \times F'_S)$.

If E and F are complete, then $E \tilde{\otimes}_\epsilon F$ is a closed subspace of the complete space $£_e(E'_S \times F'_S)$.

Proof: We know that $E \otimes F$ is algebraically imbedded in $£_e(E'_S \times F'_S)$, $x \otimes y$ is that element of $£_e(E'_S \times F'_S)$ defined by $x \otimes y\ (u\ v) = (u\ x)\ (v\ y)$, we must show that the toplogies agree.

The neighbourhoods of 0 in $E \otimes_\epsilon F$ are given by

$(G_1 \otimes G_2)^0 = \{B: |B (G_1 \otimes G_2)| \le 1\}$,

and the neighbourhoods of 0 in $£_e(E'_S \times F'_S)$ are given by

$\{B : |B (G_1 \times G_2)| \le 1\}$,

So the toplogies agree.

The second statement of the proposition is now clear.

Proposition 5: If E and F are complete, $E \tilde{\otimes}_\epsilon F$ is a closed subspace of $\mathcal{L}_e(E'_k , F)$.

Proof : By help of this

$£_e(E'_S \times F'_S) = \mathcal{L}_e(E'_k , F)$.

So, this proposition is just a statement of proposition 4.

Thus, from above definitions, facts, Remarks, and Propositions have the main result as follows:

II. Main Result

Theorem: An injective tensor product of two separated convex spaces is isomorphic to a subspace of an injective tensor product of products of two suitable families of Banach spaces.

Proof: Let E and F be two separated convex spaces. Let $(p_i)_{i\in I}$ and $(q_j)_{j\in J}$ be families of semi-norms which define the topologies of E and F respectively. For each $i\in I$ and $j\in J$

Let

$L_i = \{x \in E : p_i(x) = 0\}$,

$M_j = \{y \in F : q_j(y) = 0\}$,

$E_i = E / L_i$, $F_j = F / M_j$.

Let each E_i be equipped with the quotient norm $\hat{P}_i$ and let each F_j be equipped with the quotient norm $\hat{q}_j$. For each $x \in E$, let x_i be its canonical image in E_i and for each $y \in F$, let y_j be its canonical image in F_j. Then the map $u : x \to (x_i)$ is an isomorphism of E into $\pi\ \hat{E}_k$ and the map $v : y \to (y_j)$ is an isomorphism of F into $\pi\ \hat{F}_j$, where $\hat{E}_i$ and $\hat{F}_j$ are Banach spaces.

We now prove that $u \otimes v$ is an isomorphism of $E \otimes\varepsilon F$ into $(\pi\ \hat{E}_i) \otimes\varepsilon (\pi \hat{F}_j)$ which will yield the desired theorem.

It is trivial to verify that $u \otimes v$ is an injective linear map. To prove that $u \otimes v$ is continuous we identify $E \otimes\varepsilon F$ with $\beta\ (E'xF')$ and $(\pi\ \hat{E}_i) \otimes\varepsilon (\pi\ \hat{F}_j)$ with $\beta\ ((\Sigma\ E'_i)_\sigma \times (\Sigma\ F'_j)_\sigma)$,

Where $\beta\ (E'_\sigma \times F'_\sigma)$ denotes the space of continuous bilinear forms on $E' \times F'$ and $\beta\ ((\Sigma\ E'_i)_\sigma \times (\Sigma\ F'_j)_\sigma)$ denotes the space of continuous bilinear forms on $(\Sigma\ E'_i)_\sigma \times (\Sigma\ F'_j)_\sigma$, and Σ denotes the external direct sum. Then the mapping $u \otimes v$ can be considered as the mapping which associates with every continuous bilinear form $B\ (x', y')$ on $E'_\sigma \times F'_\sigma$, the continuous bilinear form:

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B (u' (t'), v' (z'))

On $(\Sigma\ E_i')_\sigma$ x $(\Sigma\ F_j')_\sigma$ where $t' \in \Sigma\ E_i'$ and $z' \in \Sigma\ F_j'$, and u', v' denote the transposes of the mapping u and v respectively. If A and B denote the equicontinuous subsets of $\Sigma\ E_i'$ and $\Sigma\ F_j'$ respectively, then u'(A) and v'(D) are equicontinuous subsets of E' and F' respectively. Hence if the absolute value of B is less than or equal to 1 on u'(A) x v'(D), then | B (u' (t'), v' (z')) | $\leq$ 1 on
A x D. Therefore, $u \otimes v$ is continuous.

We next show that $(u \otimes v)^{-1}$ is continuous on $(u \otimes v)$ $(E \otimes_\varepsilon F)$. For this purpose, let U and V denote any balanced convex neighbourhoods of 0 in E and F respectively. We select neighbourhoods U_1, V_1 of 0 in $\pi\ \hat{E}_i$ and $\pi\ \hat{F}_j$ respectively such that

$U_1 \cap u\ (E) \subset u(U)$,

$V_1 \cap v\ (F) \subset v(V)$,

Taking the plars we get

$u'^{-1}\ (U^0) \subset U_1^0 + \ker u'$,

$v'^{-1}\ (V^0) \subset V_1^0 + \ker v'$,

Hence $U^0 \subset u'(U_1^0)$, $V^0 \subset v'(V_1^0)$. Therefore, the bilinear form B (u' (t'), v' (z')) converges to 0 uniformly on U_1^0 x V_1^0.

Hence B (x', y') must converges to 0 uniformly on U^0 x V^0. Thus $u \otimes v$ is an isomorphism of $E \otimes_\varepsilon F$ into $(\pi\ \hat{E}_i) \otimes_\varepsilon (\pi\ \hat{F}_j)$.

This completes the proof of the theorem.

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