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An Eigenvalue Problem Associated With a System of n Simultaneous Differential Equations

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***Abstract*-- In this paper, an n x n matrix differential operator L has been defined leading to a system of n differential equations. We discussed the condition for self Adjointness for a boundary value problem, an important lemma, Green's formula for a boundary value problem with the concerned eigenvalue problem associated with matrix differential operator L.**

Keyword: **Eigenvalue, Boundry Condition, Boundary value Problem.**

I. INTRODUCTION :

1.1 Let L denote the matrix operator

$$L \equiv \begin{bmatrix} \frac{d^2}{dx^2} & -A_{11} & -A_{12} & -A_{13}\ldots\ldots & -A_{1n} \\ -A_{21} & \frac{d^2}{dx^2} & -A_{22} & -A_{23}\ldots\ldots & -A_{2n} \\ -A_{31} & -A_{32} & \frac{d^2}{dx^2} & -A_{33}\ldots\ldots & -A_{3n} \\ ----- & ----- & ----- & ----- & ----- \\ -A_{n1} & -A_{n2} & -A_{n3}\ldots\ldots \frac{d^2}{dx^2} & & -A_{nn} \end{bmatrix}$$

....................(1.1)

$\Psi \equiv \Psi(x)$ a vector having n components represented as a column matrix.

$$\Psi \equiv [\Psi_1(x),\ldots\ldots\Psi_n(x)]^T = [\Psi_1,\ldots\ldots\Psi_n]^T$$

And

$$A = \left[A_{ij} \right]_{n\times n}$$

A symmetric matrix of order n, where A_{ij} are real-valued functions of *x*,
Then *n* simultaneous differential equations

$$\left.\begin{array}{l}\dfrac{d^2\Psi_1}{dx^2}-A_{11}\Psi_1-A_{12}\Psi_2..........-A_{1n}\Psi_n=\lambda\ \Psi_1\\ \dfrac{d^2\Psi_2}{dx^2}-A_{21}\Psi_1-A_{22}\Psi_2..........-A_{2n}\Psi_n=\lambda\Psi_2\\ ------------------------\\ \dfrac{d^2\Psi_n}{dx^2}-A_{n1}\Psi_1-A_{n2}\Psi_2..........-A_{nn}\Psi_n=\lambda\Psi_n\end{array}\right\}$$

............(1.2)

are equivalent to matrix equation

$$(L-\lambda I)\Psi=0 \qquad ....(1.3)$$

where λ is real or complex parameter

The system (1.2) or (1.3) is a homogeneous system.

The non-homogeneous system associated with (1.2) or (1.3) is given by

$$\left.\begin{array}{l}\dfrac{d^2\Psi_1}{dx^2}-A_{11}\Psi_1-A_{12}\Psi_2.........-A_{1n}\Psi_n-\lambda\Psi_1\quad=-f_1\\ \dfrac{d^2\Psi_2}{dx^2}-A_{21}\Psi_1-A_{22}\Psi_2........-A_{2n}\Psi_n-\lambda\Psi_2\quad=-f_2\\ --------------------------\\ \dfrac{d^2\Psi_n}{dx^2}-A_{n1}\Psi_1-A_{n2}\Psi_2.........-A_{nn}\Psi_n-\lambda\Psi_n\quad=-f_n\end{array}\right\}$$

..............(1.4)

Where f_1, f_2, ... f_n are real-valued functions of x in $a \le x \le b$.

In matrix notation, (1.4) is equivalent to

$$(L-\lambda I)\Psi=-f \qquad ....(1.5)$$

Where

$$f\equiv[f_1,\ f_2,..............f_n]^T$$

We assume that A_{ij} are real-valued and continuous in $a \le x \le b$.

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Let the accent as usual denote differentiation with respect to x. We define the boundary conditions to be satisfied by the vector

$$\Psi \equiv [\Psi_1, \Psi_2, \ldots\ldots\ldots \Psi_n]^T$$

at $x = a$ and $x = b$ respectively by

$$(a_{j1}\Psi_1 + a_{j2}\Psi_1') + (a_{j3}\Psi_2 + a_{j4}\Psi_2')$$
$$+ (a_{j5}\Psi_3 + a_{j6}\Psi_3') + \ldots\ldots + (a_{j2n-1}\Psi_n + a_{j2n}\Psi_n')$$

= 0, j = 1, 2, ..., n.

................(1.6)

and

$$(b_{j1}\Psi_1 + b_{j2}\Psi_1') + (b_{j3}\Psi_2 + b_{j4}\Psi_2')$$
$$(b_{j5}\Psi_3 + b_{j6}\Psi_3') + \ldots\ldots + (b_{j2n-1}\Psi_n + b_{j2n}\Psi_n')$$

= 0, j = 1,2,................., n

................(1.7)

where

(a) a_{jk} and b_{jk} j = 1, 2,................, n; k = 1, 2, ...2n are real-valued constants;

(b) The trivial case of $a_{jk} = 0$ for j = 1, 2.................,n.

k = 1, 2, ..., 2n and $b_{jk} = 0$ for j = 1, 2,............., n,

k = 1, 2, ..., 2n is excluded.

(c) The sets $\{a_{i1}, a_i, \ldots, a_{in}\}$

i = 1, 2, ..., n

are linearly independent, and the sets

$\{b_{i1}, b_{i2}, \ldots, b_{in}\}$ i = 1, 2,.............., n

are linearly independent.

(d)
$$(a_{j1}\, a_{k2} - a_{j2}\, a_{k1}) + (a_{j3}\, a_{k4} - a_{j4}\, a_{k3})$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots + (a_{j2n-1}\, a_{k2n} - a_{j2n}\, a_{k2n-1}) = 0$$
$$1 \le j, k \le n,$$

.... (1.8)

$$(b_{j1}\, b_{k2} - b_{j2}\, b_{k1}) + (b_{j3}\, b_{k4} - b_{j4}\, b_{k3})$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots + (b_{j2n-1}\, b_{k2n} - b_{j2n}\, b_{k2n-1}) = 0$$

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and $1 \le j, \quad k \le n$ ....(1.9)

1.2 The boundary conditions (1.6) and (1.7) can be represented in the alternative 'Kodaira form' as given below :

$$\left[\theta(x,\lambda)\phi_j(a/x,\lambda)\right]_{x=a} = \left[\theta\,\phi_j\right](a) =, (j=1,2,......,n)$$
.... (1.10)

And

$$\left[\theta(x,\lambda)\phi_k(b/x,\lambda)\right]_{x=b} = \left[\theta\,\phi_k\right](b) = 0, (k=n+1, n+2, ..., 2n)$$
.... (1.11)

Where $\phi_j\,(a/x,\lambda)\,and\,\phi_k\,(b/x,\lambda)$ are called boundary condition vector at *a* and *b* respectively.

$$\phi_j\,(a/x,\lambda) = [u_{j1}\,(a/x,\lambda), ..., u_{jn}(a/x,\lambda)]^{T}$$

$$(j=1,2,......n)$$

And $\phi_k\,(b/x,\lambda) = [u_{k1}\,(a/x,\lambda), ..., u_{kn}(a/x,\lambda)]^{T}$

$$(k=n+1, ..., 2n)$$ ....(1.12)

And $\Theta(x,\lambda) = [\Theta_1(x,\lambda), \Theta_2(x,\lambda) \,........\, \Theta_n(x,\lambda)]^T$ be a vector satisfying (1.6) and (1.7)

We shall be concerned only with the vectors $\Theta(x,\lambda)$ which are solutions of (1.3). In this case we may choose the value of λ in $\phi_j\,(a/x, \lambda)$ and $\phi_k\,(b/x, \lambda)$ to be the same as in $\Theta(x, \lambda)$ and then (1.10) and (1.11) may be written as

$[\Theta(x, \lambda)\ \phi_j\,(a/x, \lambda)] = 0;\ j = 1, 2, ...n$ .....(1.13)

and

$[\Theta(x, \lambda)\ \phi_k\,(b/x, \lambda)] = 0;\ k = n+1, n+2, ... 2n$ .....(1.14)

Now it is easily seen that conditions (1.9) and (1.10) are

equivalent to $[\phi_i\ \phi_j] = 0,\ 1 \le i,\ j \le n,$

..............(1.15)

And

$$[\phi_k\ \phi_l] = 0 \qquad n+1 \le k, \qquad 1 \le 2n,$$ ....(1.16)

respectively. We note that $\left[\phi_i\ \phi_j\right]\,and\,\left[\phi_k\ \phi_l\right],$

$1 \le i,\ j \le n, n+1 \le k, 1 \le 2n,$ and are independent of *x* and λ

1.3 Let $\phi_j = \phi_j(x) = [\phi_{j1}(x), \phi_{j2}(x), \ldots\ldots \phi_{jn}(x)]^T$

$\phi = \phi_k(x) = [\phi_{k1}(x), \phi_{k2}(x), \ldots\ldots \phi_{kn}(x)]^T$

Where ϕ_j & ϕ_k are functions of x, be two given vectors. Then the bilinear concomitant of the two vectors ϕ_j and ϕ_k is denoted by $[\phi_j\ \phi_k]$ and is defined as

$$[\phi_j\ \phi_k] = (\phi'_{j1}\,\phi_{k1} - \phi_{j1}\,\phi'_{k1}) + (\phi'_{j2}\,\phi_{k2} - \phi_{j2}\,\phi'_{k2}) + \ldots\ldots\ldots\ldots\ldots$$
$$+ (\phi'_{jn}\,\phi_{kn} - \phi_{jn}\,\phi'_{kn}) \qquad \ldots(1.17)$$

We now prove some of the important properties of the bilinear concomitant

(a) It is clear that

(i) $[\phi_j\ \phi_j] = 0$

(ii) $[\phi_j\ \phi_k] = -[\phi_k\ \phi_j]$

(iii) $[\phi_j\ (k_1\,\phi_l + k_2\,\phi_m)] = k_1\,[\phi_j\ \phi_l] + k_2\,[\phi_j\ \phi_m]$

where k_1 and k_2 are constants.

(b) If ϕ_j and ϕ_k are solutions of (1.2) or (1.3) for the same value of λ.

$[\phi_j\ \phi_k]$ is independent of x and is a function of λ alone.

(c) Let ϕ^T denote the transpose of ϕ then

$\phi^T_k\,(L\,\phi_j) - \phi^T_j\,(L\phi_k) = (\phi_{ki}\,''_{j1} - \phi_{j1}\,''_{k1}) + (\phi_{k2}\,\phi''_{j2} - \phi_{j2}\,\phi''_{k2}) + \ldots\ldots\ldots + (\phi_{kn}\,\phi''_{jn} - \phi_{jn}\,\phi''_{kn}) + (\phi_{k1}\,\phi'_{j1} - \phi_{j1}\,\phi'_{k1}) + (\phi_{k2}\,\phi'_{j2} - \phi_{j2}\,\phi'_{k2}) + \ldots\ldots$

$+ (\phi_{kn}\,\phi'_{jn} - \phi_{jn}\,\phi'_{kn})$

i.e. $\phi^T_k\,(L\,\phi_j) - \phi^T_j\,(L\,\phi_k) = [\phi_j\,\phi_k]^1 \qquad \ldots(1.18)$

II. Now We Prove The Following Lemma

Let

$$f_r = [f_{r1}, \quad f_{r2}, \ldots\ldots\ldots\ldots, f_{rn}]^T$$

$$g_s = [g_{s1}, \quad g_{s2}, \ldots\ldots\ldots\ldots, g_{sn}]^T$$

(r, s = 1,2 ....2n + 1)

be vector having first-order derivatives. Then

$$\det_{1\le r,\, s\le 2n+1} [f_r g_s] = 0 \qquad \text{---------- (2.1)}$$

we see that

$$\begin{bmatrix} [f_1\ g_1] & [f_1\ g_2] \ldots\ldots\ldots\ldots\ldots\ldots [f_1\ g_{2n+1}] \\ [f_2\ g_1] & [f_2\ g_2] \ldots\ldots\ldots\ldots\ldots [f_2\ g_{2n+1}] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [f_{2n+1}\ g_1] & [f_{2n+1}\ g_2] \ldots\ldots\ldots [f_{2n+1}\ g_{2n+1}] \end{bmatrix} =$$

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$$\begin{bmatrix} f'_{11}\cdots\cdots\cdots & f'_{1n} & f_{11}\cdots\cdots\cdots & f_{1n} & 0 \\ f'_{21}\cdots\cdots\cdots & f'_{2n} & f_{21}\cdots\cdots\cdots & f_{2n} & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f'_{2n+11}\cdots\cdots\cdots & f'_{2n+1n} & f_{2n+11}\cdots\cdots\cdots & f_{2n+1n} & 0 \end{bmatrix}$$

$$\begin{bmatrix} g_{11} & g_{21}\cdots\cdots\cdots & g_{2n+1\ 1} \\ g_{12} & g_{22}\cdots\cdots\cdots & g_{2n+1\ 2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ g_{1n} & g_{2n}\cdots\cdots\cdots & g_{2n+1\ n} \\ -g'_{11} & -g'_{21}\cdots\cdots\cdots & -g_{2n+1\ 1} \\ -g'_{12} & -g'_{22}\cdots\cdots\cdots & -g_{2n+1\ 2} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ -g'_{1n} & -g'_{2n}\cdots\cdots\cdots & -g'_{2n+1\ n} \\ 0 & 0\cdots\cdots\cdots & 0 \end{bmatrix}$$

Taking deter nant both sides, we get (2.1)

We now prove the following theorem :

Theorem 2.1. : Let any two vectors

$$F(x)=[F_1(x),\ F_2(x),\dots\dots\dots,F_n(x)]^T$$

$$G(x)=[G_1(x),\ G_2(x),\dots\dots\dots,G_n(x)]^T$$

satisfy the boundary conditions at $x=a$ and $x=b$ respectively.

Then

$$[F(x)\ \ G(x)]=0$$

both at x = a and x = b

Proof : First we prove that $[F(x)\ \ G(x)]=0$

Since $F(x)$ and $G(x)$ satisfy at $x = a$ the relation (1.6) and (1.8) are satisfied, we have from (1.10) and (1.15)

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$$\left.\begin{array}{l}\left[F\phi_j\right](a)=0,\\ \\ \left[G\phi_j\right](a)=0\end{array}\right\}$$

(j = 1, 2,.................., n) ....(2.2)

And $\left[\phi_i\ \phi_j\right]=0, \quad 1\le i, \quad j\le n$ ....(2.3)

Where $\phi_j=\phi_j(a/x,\lambda),$ j = 1, 2,..........n are

boundary condition vectors at $x = a$

Now in (2.1) put

$$f_r=g_r=\phi_r=\phi_r(a/x,\lambda), \qquad 1\le r\le n$$

$$f_r=g_r=\phi_r=\phi_r(a/x,\lambda), \qquad n+1\le r\le n$$

$$f_{2n+1}=F, g_{2n+1}=G,$$

where $\phi_{n+1},=\phi_{n+2},..............,\phi_{2n}$ are any vectors,

Thus we get

$$\begin{vmatrix}[\phi_1\ \phi_1]\ .....[\phi_1\ \phi_n]\ \ [\phi_1\ \phi_{n+1}]...[\phi_1\ \phi_{2n}]\ [\phi_1\ G]\\ ....\\ [\phi_n\ \phi_1]\ ....[\phi_n\ \phi_n]\ \ [\phi_n\ \phi_{n+1}]..............[\phi_n\ \phi_{2n}]\ [\phi_n\ G]\\ [\phi_{n+1}\ \phi_1]...[\phi_{n+1}\ \phi_n]\ \ [\phi_{n+1}\ \phi_{n+1}].......[\phi_{n+1}\ \phi_{2n}]\ [\phi_{n+1}\ G]\\ ....\\ [\phi_{2n}\ \phi_1]\ ....[\phi_{2n}\ \phi_n]\ \ [\phi_{2n}\ \phi_{n+1}]........[\phi_{2n}\ \phi_{2n}]\ [\phi_{2n}\ G]\\ [F\ \phi_1]\ ....[F\ \phi_n]\ \ [F\ \phi_{n+1}]..................[F\ \phi_{2n}]\ [F\ G]\end{vmatrix}=0 \qquad ....(2.4)$$

Since $\phi_{n+1},\ \phi_{n+2},............,\phi_{2n}$ are arbitrary vectors, we can so choose them that

$$\begin{vmatrix}[\phi_1\ \phi_{n+1}]..........\ [\phi_1\ \phi_{2n}]\\ [\phi_2\ \phi_{n+1}]..........\ [\phi_2\ \phi_{2n}]\\ ....\\ [\phi_n\ \phi_{n+1}]........\ [\phi_n\ \phi_{2n+1}]\end{vmatrix}\ne 0 \qquad ....(2.5)$$

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For all λ.

Expanding (2.4) by its last row and using (2.2) and (2.3),

we get at $x = a$

$$[F\,G] = \begin{vmatrix} [\phi_1 \;\; \phi_{n+1}] \ldots\ldots\ldots\ldots [\phi_1 \;\; \phi_{2n}] \\ [\phi_2 \;\; \phi_{n+1}] \ldots\ldots\ldots\ldots [\phi_2 \;\; \phi_{2n}] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [\phi_n, \;\; \phi_{n+1}] \ldots\ldots\ldots [\phi_n \;\; \phi_{2n+1}] \end{vmatrix} = 0$$

From which we obtain

$[F\ G] = 0$ at $x = a$,

Using (2.5),

Similarly we can prove $[F\ G] = 0$ at $x = b$.

III. Green's Formula For Boundary Value Problem

Let $F = F(x) = [f_1, f_{2,} \ldots\ldots\ldots\ldots\ldots, f_n]^T$

$$F = G(x) = [g_1, g_{2,} \ldots\ldots\ldots\ldots\ldots, g_n]^T$$

Where f_1, f_2, ... f_n, g_1, g_2, ...g_n are functions of x, be two vectors having continuous derivatives of the second order.

Then we have by (1.18)

$$G^T(L\ \ F) - F^T\ (L\ \ G) = [F\ \ G]'.$$

Thus

$$\int_a^b \left(G^T L\ F - F^T\ L\ G\right) dx = [F\ G]_a^b$$

$$= [F\ \ G](b) - [F\ \ G](a) \qquad \ldots\ldots\ldots(3.1)$$

This is the Green's formula for our boundary value problem. Now if F and G are such that

$$[F\ \ G](b) - [F\ \ G](a) = 0, \qquad \ldots\ldots\ldots\ldots(3.2)$$

Then

$$\int_a^b G^T L\ F\ dx = \int_a^b F^T L\ G\ dx\ , \qquad \ldots\ldots\ldots\ldots(3.3)$$

The Self adjointness condition

Let F and G be the solutions of (1.3) and satisfy the boundary conditions (1.6) and (1.7) at $x = a$ and $x = b$ respectively. If the conditions (1.8) and (1.9) are satisfied then by theorem 2.1.1

$$[F \;\; G](b) = [F \;\; G](a) = 0,$$

Hence our boundary value problem given by (1.3), (1.6) and (1.7) is self adjoint if condition (1.8) and (1.9) hold.

IV. CONCLUSION

In this way an important lemma 2.1, theorem 2.1.1, Green's formula for a boundary value problem 3.1 have been discussed. Hence our boundary value problem given by (1.3), (1.6) & (1.7) is self adjoint if conditions (1.8) and (1.9) hold.

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