EXPLORING RESEARCH AND INNOVATIONS

International Journal of Emerging Technology and Advanced Engineering
Website: www.ijetae.com (ISSN 2250-2459, ISO 9001:2008 Certified Journal, Volume 7, Issue 4, April 2017)

New Trends on Duality Theory in Hilbert Space

C. S. Prasad[1], A. Kumar[2] A. K. Singh[3], U. K. Srivastava[4]

Department of Mathematics

[1] L.N.T. College, Muzaffarpur- 842002, B.R.A. Bihar University,Muzaffarpur- 842001, Bihar , India.
[2]P.R.R.D. College, Bairgania, Sitamarhi – 843313, B.R.A. Bihar University,Muzaffarpur- 842001, Bihar , India.
[3] S.N.S. College, Hajipur (Vaishali)- 844101, B.R.A. Bihar University, Muzaffarpur – 842001, Bihar , India.
[4]R.S.S. College, Chochahan, P.O. – Aniruddh Belsar, Dist.- Muzaffarpur- 844111, B.R.A. Bihar University, Muzaffarpur- 842001, Bihar , India.

***Abstract*-- This paper presents the study of New Trends on Duality theory in Hilbert Space . Here we consider a Linear functional on a Complex Hilbert Space H, which is bounded or continuous and by defining the norm of a bounded linear functional φ, it is proved in this paper that , if y Є H, $\varphi_y(x) = \langle y , x \rangle$, ∀ x Є H is a bounded linear functional on H with $\| \varphi_y \| = \| y \|$**

***Keywords*-- Hilbert space, Duality theory, Norm, Riesz Representation, Isometry.**

I. INTRODUCTION

Hall, M.(1) and Kothe (2,3) are the pioneer worker of the present area. In fact, the present work done by Wong, Yau-Chuen (8), Srivastava et al.(4), Srivastava et al.(5), Kumar et al.(6), and Srivastava et al.(7). In this paper we have studied duality theory in Hilbert Space.

Here we use the following definitions, notations, and Fundamental ideas :

Definition 1: Any projection associated with a direct sum decomposition of a projection on a Linear space X is a linear map $P:X \to X$ such that $P^2 = P$

Definition 2: An orthogonal projection on a Hilbert space H is also a Linear mapping $P:H \to H$ satisfying $P^2 = P$, $\langle Px,y\rangle = \langle x , Py\rangle$ for all $x, y \in H$.

"An orthogonal projection is necessarily bounded."

Theorem 1 : Let X be a linear space,

(i) If $P:X \to X$ is a projection then $X = \text{ran } P \oplus \ker P$

(ii) If $X = M \oplus N$ where M and N are Linear subspaces of X then there is a projection $P:X \to X$ with ran P = M and ker P = N.

Proof:

For (i) We show that $x \in \text{ran } P$ if $x = Px$

If $x = Px$ then clearly $x \in \text{ran } P$

If $x \in \text{ran } P$ then $x = Py$ for some $y \in x$

And since $P^2 = P$ which follows that $Px = P^2y = Py = x$

If $x \in \text{ran } P \cap \ker P$ then $x = Px$ & $Px = 0$

So $\text{ran } P \cap \ker P = \{0\}$. If $x \in X$ then

We have $x = Px + (x- Px)$; where $Px \in \text{ran } P$ and $(x - Px) \in \ker P$.

Since $P (x- Px) = Px - P^2x = Px - Px = 0$

Thus $X = \text{ran } P \oplus \ker P$. ………………………(1.1)

Now for (ii)

We consider if $X = M \oplus N$ then $x \in N$ has unique decomposition $x = y+z$ with

$y \in M$ & $Z \in N$ and $Px = y$ defines the required Projection .

In particular, in orthogonal subspaces while using Hilbert Space, let us suppose that M is a closed subspace of Hilbert Space H then by well known property we have $H = M \oplus M^{\perp}$. We call the projection of H on to M along $M^{\perp}$ the orthogonal projection of H on to M.

If $x= y+z$ and $x_1 = y_1 + z_1$ where $y, y_1 \in M$ and $z, z_1 \in M^{\perp}$ then by orthogonality of M and $M^{\perp} \Rightarrow \langle Px, x_1\rangle = \langle y, y_1 +z_1\rangle = \langle y, y_1\rangle = \langle y+z, y_1\rangle$

$= \langle x, Px_1\rangle$ …………… (1.2)

Which states that an orthogonal projection is self Adjoint. We show the properties (1.1) and (1.2) characterize orthogonal projections with Defn-2.

IJETAE EXPLORING RESEARCH AND INNOVATIONS

International Journal of Emerging Technology and Advanced Engineering

Website: www.ijetae.com (ISSN 2250-2459, ISO 9001:2008 Certified Journal, Volume 7, Issue 4, April 2017)

Lemma :- If P is a non zero orthogonal projection then $\| P\| = 1$.

Proof : - If $x \in H$ and $Px \neq 0$ then by Cauchy Schwarz inequality ,

$$\| Px \| = \frac{<Px, Px>}{\| Px \|} = \frac{<x, P^2x>}{\| Px \|} = \frac{<x, Px>}{\| Px \|} \leq \| x\|$$

Therefore $\| P \| \leq 1$. If $P \neq 0$ then there is an $x \in H$ with $Px \neq 0$ and $\| P(Px) \| = \| Px \|$ so that $\| P\| \geq 1$.

Thus, the Orthogonal Projection P and closed subspace M of H such that ran P = M will must obey one –one correspondence, then the kernel of Orthogonal Projection is the Orthogonal Complement of M.

Example .1 – The space L^2 (R) is the Orthogonal direct sum of space M of even functions and the space N of odd functions .

The Orthogonal Projection P and Q of H onto M and N, respectively are given by

$$Pf(x) = \frac{f(x) + f(-x)}{2}, \quad Q f(x) = \frac{f(x) – f(-x)}{2}$$

Where I- P = Q .

Example 2 – If $H = R^n$, the orthogonal projection Pu in the direction of a unit vector u has the rank one matrix Uu^T. The component of a vector X in the direction U is

$$PuX = (u^T X) u$$

Example 3 : - If $H = L^2$ (T) is the space of 2π-Periodic function and $u = 1/\sqrt{2\pi}$ is the constant function with norm one, then the Orthogonal projection P_u maps a function to its mean :

$$Pu f = < f > ,\text{Where } < f > = 1/2\pi \int_0^{2\pi} f(x)\, dx$$

The corresponding Orthogonal decomposition, $f(x) = < f > + f^{/}(x)$ decompose a function in to a constant mean part $< f >$ and a fluctuating part $f^{/}$ with zero mean .

Example: 4 Suppose $H = L^2$ (T), then for each $n \in Z$ the functional

$\varphi_n : L^2(T) \to C$, $\varphi_n (f) = 1 /\sqrt{2\pi} \int_T f(x)\, e^{-inx} dx$ that maps a function to its nth Fourier coefficient is a bounded linear functional. We also have $\| \varphi_n \| = 1$ for every $n \in Z$.

Proposition : (a) A Linear functional on a Complex Hilbert space H is a Linear map from H to C. A Linear functional φ is bounded or continuous, if there exists a constant M such that $| \varphi (x) | \leq M \| x \|$ for all $x \in H$.

The norm of bounded linear functional φ is

$$\| \varphi \| = \sup_{\| x \| = 1} |\varphi (x) |$$

If $y \in H$ then $\varphi_y (x) = < y, x >$ is a bounded Linear functional on H, with

$\| \varphi_y \| = \| y\|$.

(b) If φ is a bounded Linear functional on a Hilbert space H, then there is a unique vector $y \in H$ such that

$\varphi (x) = < y, x>$ for all $x \in H$

Thus, from above definitions, theorem, Lemma, examples, which shows the Main result :

Main result :- Theorem 2 :-(Riesz Representation) :- If φ is a bounded linear functional on a Hilbert space H, then there is a Unique Vector $y \in H$ such that

$\varphi_y (x) = < y,x >$, for all $x \in H$ with $\| \varphi_y \| = \|y\|$.

Proof :- . If $\varphi = 0$, then $y = 0$, so we suppose that $\varphi \neq 0$. In that case , $\ker \varphi$ is a proper closed subspace of H. and , it implies that , there is a nonzero vector

$z \in H$ such that $z \perp \ker\varphi$. We define a linear map $P : H \to H$ by

$Px = \varphi (x) /\varphi (z) .z$

Then $P^2 = P$, so Theorem 1 implies that , H = ran P $\oplus$ kerP. Moreover,

ran P $=\{\alpha z | \alpha \in C\}$, kerP = $\ker\varphi$

So that ran P $\perp$ ker P. It follows that P is an orthogonal projection, and

$H =\{\alpha z|\alpha \in C\} \oplus \ker\varphi$ is an orthogonal direct sum. We can therefore write

$x \in H$ as $x = \alpha z + n$, $\alpha \in C$ and $n \in \ker\varphi$.

Taking the inner product of this decomposition with z, we get

$\alpha = \langle z, x \rangle / \| z \|^2$, and evaluating φ on $x = \alpha z + n$, we find that

$\varphi(x) = \alpha\, \varphi(z)$.

The elimination of α from these equations, and a rearrangement of the result,

yields $\varphi(x) = \langle y, x \rangle$, where $y = \varphi(z)/\| z \|^2 . z$.

Thus, every bounded linear functional is given by the inner product with a fixed vector. We have already, seen that $\varphi_y(x) = \langle y, x \rangle$ defines a bounded linear functional on H for every $y \in H$. To prove that there is a unique y in H associated with a given linear functional, suppose that $\varphi_{y1} = \varphi_{y2}$. Then $\varphi_{y1}(y) = \varphi_{y2}(y)$. When $y = y_1 - y_2$, which implies that

$\| y_1 - y_2 \|^2 = 0$, so $y_1 = y_2$.

The Map $J : H \to H^*$ given by $J_y = \varphi_y$, therefore identifies a Hilbert space H with its dual space H*. The norm of φ_y is equal to the norm of y, so j is an isometry. In this case of complex

Hilbert spaces, J is antilinear, rather than linear, because $\varphi_{\lambda y} = \lambda \varphi_y$. Thus, Hilbert spaces are self – dual, meaning that H and H* are isomorphic as Banach spaces, and anti-isomorphic as Hilbert spaces. Thus Hilbert spaces are special in this respect. This completes the proof of the main result.

Acknowledgement:

The authors are thankful to Prof.(Dr.) S.N. Jha, Ex. Head, Prof. (Dr.) P.K. Sharan, Ex. Head, and Prof.(Dr.) Ganesh Kumar, Ex. Head, University Department of Mathematics, B.R.A.B.U. Muzaffarpur, Bihar, India and Prof.(Dr.) T.N. Singh, Ex. Head, Ex. Dean(Science) and Ex. Chairman, Research Development Council, B.R.A.B.U. Muzaffarpur, Bihar, India for extending all facilities in the completion of the present research work.

REFFERENCES

[1] Hall, M. (1959) : J.The theory of groups, Mac Millan , New York.

[2] Kothe, G. (1969) : Topological Vector spaces, Springer-Verlag, I, New York.

[3] Kothe, G. (1979) : Topological Vector spaces, Springer-Verlag, II, New York.

[4] Srivastava, U.K., Kumar, N., Kumar, S., and Singh, T.N.(2012) : Proc. Of Math Soc., B.H.U. Vol. 28 , pp 25-28.

[5] Srivastava, U.K., Kumar, N., and Haque, M. (2014) : International Journal of Emerging Technology and Advanced Engineering, Vol.4(9), pp 325-326.

[6] Kumar Nirmal, Prasad, C.S., Shahabuddin, Md., and Srivastava, U.K.,(2016) : International Journal of Emerging Technology and Advanced Engineering, Vol.6(10), pp 149- 152.

[7] Srivastava, U.K.,Pratap, L.R., Singh, A.K., and Shahabuddin, Md.,(2016) : International Journal of Emerging Technology and Advanced Engineering, Vol.6(12), pp 213-215.

[8] Wong, Yau-Chuen.(1969) : Order Infrabarreled Riesz Spaces, Math. Ann., Vol.183, pp 17-32