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Green's Equivalence Relations on the Multiplicative Semigroup Z_n

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***Abstract*— The Multiplicative Semigroup Z_n has a vital role in the studies of computer scientists. This paper present the various characteristics of this structure. The rank of Z_n, the characteristics of regular Z_n, the idempotents in Z_n, Green's relations on Z_n and the D-classes of Z_n are the topics under discussion. It is already known that the Multiplicative semigroup Z_n is regular iff n is square free. This paper is expected to be helpful for a comparative study between regular and non-regular Z_n.**

***Keywords*— Multiplicative semigroup integers modulo n, rank of a semigroup, regular semigroup, idempotent elements of a semigroup, Euler function, Green's relations, Green's equivalence classes.**

I. INTRODUCTION

Z_n is a group under $+_n$ and is a monoid under $\times_n$. If n is a prime, then, Z_n is a group under $\times_n$. For our discussion, Z_n is the semigroup under $\times_n$.[2,3] Already it was observed that Z_n is a regular semigroup iff n is square free.[9] Since Z_n is a commutative semigroup, the Green's equivalence relations on Z_n coincides. Then obviously each D-class of Z_n consists of only one cell in its egg-box diagram. Also regular Z_n is a best known subclass of intersection of locally inverse semigroups and E-solid semigroups. This paper mainly concentrates on the total number of distinct D-classes of Z_n.

II. PRELIMINARIES

Let S be a semigroup. The set of all idempotent elements of S is denoted by E(S). Z_n is an E-semigroup as $E(Z_n)$ forms a subsemigroup of Z_n . An element a of S is called regular if there exists x in S such that $axa = a$.[1,4,5] The set of all regular elements of S is denoted by Reg(S). The semigroup S is called regular if all its elements are regular. S is regular means Reg(S) = S. An element a' is an inverse of a if $aa'a = a$ and $a'aa' = a'$. An element with an inverse is necessarily regular.

Also every regular element has an inverse; if there exists x ∈ S such that $axa = a$ then define, $a' = xax$ and then $aa'a = axaxa = axa = a$ and $a'aa' = xaxaxax = xaxax = xax = a'$. An element a may have more than one inverse. The set of inverses of an element is denoted by $V(a)$. If every element of a regular semigroup S has a unique inverse, then S is called an inverse semigroup. Obviously, regular Z_n is an orthodox semigroup.

Theorem 2.1 [9] Consider $(Z_n, \times_n)$, the multiplicative semigroup of integers modulo 'n, where $Z_n = \{\bar{0}, \bar{1}, \ldots \overline{(n-1)}\}$, then for $x \in Z_n$, $\bar{x}$ is a regular element of Z_n iff x and $\frac{n}{(n,x)}$, where (n, x) is the g.c.d of n and x, are relatively prime.

Corollary 2.1.1. [9] The multiplicative semigroup Z_n is a regular semigroup iff n is square free.

Theorem 2.2 [10,11,12] If (a, b) = 1 and $n = ab$, then any idempotent in Z_n has the form $a^{\varphi(b)}$.

Corollary 2.2.1[10, 11,12] Let the prime factorization of n be $n = p_1^{n_1} p_2^{n_2} p_3^{n_3} \ldots p_m^{n_m}$, the prime factorization consists of m factors, then the number of idempotents in Z_n is 2^m.

A set K is a generating set of a semigroup S if S contains all possible products of elements of K. Rank of S denote the minimum cardinality of a generating set. Here we introduce a formula for rank of Z_n in the later section.

If a is an element of a semigroup S, the smallest left ideal of S containing a is $Sa \cup \{a\}$, which is denoted by S^1a and is called the principal left ideal generated by a. Similarly, we can define the principal right ideal generated by a as $aS^1 = Sa \cup \{a\}$.

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Also the principal two-sided ideal of S generated by a is S^1aS^1 . Using the ideals mentioned above J. A. Green(1951), introduced five equivalence relations on a semigroup S which are denoted by L, R, H, D and J. Let $a, b \in S$, then

(i) aLb if and only if $S^1a = S^1b$

(ii) aRb if and only if $aS^1 = bS^1$

(iii) aJb if and only if $S^1aS^1 = S^1bS^1$

(iv) $H = L \cap R$

(v) $D = L \vee R$

The corresponding equivalence classes of an element $a \in S$ are denoted by L_a, R_a, H_a, J_a and D_a respectively.[1] A semigroup S is said to be E-solid if for all $e, f, g \in E(S)$ satisfying $eLfRg$, then there exists an idempotent $h \in S$ such that $eRhLg$. Z_n is an E-solid semigroup. In Z_n, $L = R = H = D = J$, each D-class of Z_n consists of only one cell in its egg-box diagram.

III. Green's Equivalence Relations On Z_n

In this paper, the set of all numbers less than n and relatively prime to n is denoted by $R[n]$. Corresponding to each factor $x \neq 1$ of n, a set is constructed which is denoted by M_x to contain all multiples of x less than n, which are not divisible by a larger factor of n. Obviously, the sets M_x are disjoint. Example: For n = 30, the set M_2 = {2, 4, 8, 14, 16, 22, 26, 28}

Lemma 3.1.1 $Z_n = R[n] \cup (\bigcup_x M_x)$.

Proof: For each factor x of n there is an M_x and all the sets M_x are disjoint by their construction. If there is an integer u which is less than n and which does not belongs to $(\bigcup_x M_x)$, then it will be relatively prime to n and hence belongs to R[n]. Thus the lemma.

Lemma 3.1.2 Rank(Z_n) is the number of factors of n .

Proof: By lemma 3.1.1. Z_n is the disjoint union $R[n] \cup (\bigcup_x M_x)$. Then a generating set K for Z_n consists of each facor of n. Thus the result.

Theorem 3.1 The number of distinct D-classes in Z_n is the number of factors of n.

OR

If $n = p_1^{m_1} p_2^{m_2} p_3^{m_3} \ldots p_k^{m_k}$, then the number of distinct D-classes in Z_n is $(m_1 + 1)(m_2 + 1)(m_3 + 1)\ldots(m_k + 1)$.

Proof: By lemma 3.1.1, $Z_n = R[n] \cup (\bigcup_x M_x)$.

Claim01: The D-class of any element in R[n] is the same as D_1. Let $t \in R[n]$ be arbitrary. $\Rightarrow (n, t) = 1 \Rightarrow \exists\ t' \in Z_n$ such that $tt' = t' t = 1$ It is needed to prove $D_t = D_1$. It is obvious that

$tZ_n \subseteq Z_n$. Let $y \in Z_n$ be arbitrary Then $y = y1 = ytt' = tyt' \Rightarrow y \in t Z_n \Rightarrow Z_n \subseteq t Z_n$. Thus $Z_n = t Z_n \Rightarrow tD1 \Rightarrow D_1 = D_t$.

Claim02: For each factor $x \neq 1$ of n, the D-class of any element in M_x is the same as D_x. Let $mx \in M_x$ where $m \neq 1$ be arbitrary. Obviously $mx Z_n \subseteq x Z_n$. To prove the converse a function f from $x Z_n$ to $mx Z_n$ is defined as, $f(xt) = mxt$, for each $t \in Zn$. Now $f(xt_1) = f(xt_2) \Rightarrow mxt_1 = mxt_2 \Rightarrow mx(t_1 - t_2) \equiv 0(modn)$. But M_x is constructed so that mx is not a factor of n. $\Rightarrow (t_1 - t_2) \equiv 0(modn) \Rightarrow t_1 \equiv t_2(modn) \Rightarrow xt_1 \equiv xt_2(modn)$. Thus f is an injective function. Shows that $x Z_n \subseteq mx Z_n$. Thus $x Z_n = mx Z_n \Rightarrow xDmx \Rightarrow D_x = D_{mx}$.

From lemma 3.1.1, claim01 and claim02 it is obvious that R[n] contributes the D-class D_1 and each M_x contributes the D-class D_x, where $x \neq 1$ is a factor of n. Thus it is concluded that the number of distinct D-classes in Z_n is the number of factors of n, provided the D-class D_0 corresponds to the factor n itself.

Corollary: 3.1.1 The number of distinct D-classes in regular Z_n is 2^m .

Proof: By Corollary 2.1.1. we have Z_n as regular iff $n = p_1p_2p_3\ldots p_m$, a product of distinct primes. Then by theorem 3. 1. there will be 2^m distinct D-classes.

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Example 3.1.1. Consider Z_{30}, an example for a regular Z_n.

We have R[30] = {1, 7, 11, 13, 17, 19, 23, 29} and the Euler Phi function φ(30) = 8.

$Z_{30} = 7Z_{30} = 11Z_{30} = 13Z_{30} = 17Z_{30} = 19Z_{30} = 23Z_{30} = 29Z_{30}$. The D-classes of all elements of R[30] are the same as D-class of 1 denoted by D_1.

Now $2Z_{30}$ = {0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28} = $4Z_{30} = 8Z_{30} = 14Z_{30} = 16Z_{30} = 22Z_{30} = 26Z_{30} = 28Z_{30}$. Shows that the D-class of 2 consists of D_2 = {2, 4, 8, 14, 16, 22, 26, 28}.

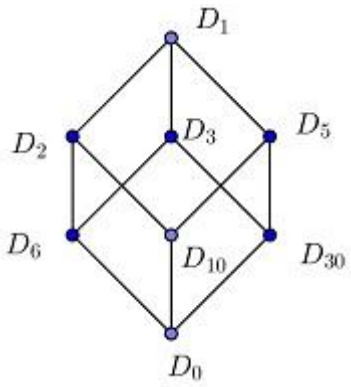
$3Z_{30}$ = {0, 3, 6, 9, 12, 15, 18, 21, 24, 27} = $9Z_{30} = 21Z_{30} = 27Z_{30}$. Shows that the D-class of 3 consists of D_3 = {3, 9, 21, 27}.

$5Z_{30}$ = {0, 5, 10, 15, 20, 25} = $25Z_{30}$ Shows that the D-class of 5 consists of D_5 = {5, 25}. $6Z_{30}$ = {0, 6, 12, 18, 24} = $12Z_{30} = 18Z_{30} = 24Z_{30}$ Shows that the D-class of 6 consists of D_6 = {6, 12, 18, 24}.

$10Z_{30}$ = {0, 10, 20} = $20Z_{30}$ Shows that the D-class of 10 consists of D_{10} = {10, 20}.

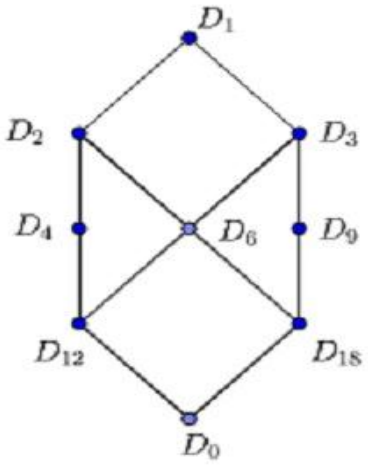
$15Z_{30}$ = {0, 15} Shows that the D-class of 15 consists of D_{15} = {15}. Now $0Z_{30}$ = {0} Shows that the D-class of 0 consists of D_0 = {0}.

Thus the distinct D-classes of Z_{30} are D_1, D_2, D_3, D_5, D_6, D_{10}, D_{15}, D_0. Corresponding to each factor of 30 there is a D-class. Totally there are 8 distinct classes. The natural partial ordering on the set of D-classes is given by the Hasse diagram,



Example 3.1.2. Consider Z_{36}, an example for a non-regular Z_n.

The distinct D-classes of Z_{36} are D_1, D_2, D_3, D_4, D_6, D_9, D_{12}, D_{18} and D_0, the Hasse Diagram of the lattice of D-classes of Z_{36} is given below:



Corollary 3.1.2 In Z_n, the number of idempotents is less than or equal to the number of distinct D-classes.

Proof: By theorem 3.1, the number of distinct D-classes in Z_n is $(m_1+1)(m_2+1)(m_3+1)....(m_k+1)$ whenever $n = p_1^{m_1} p_2^{m_2} p_3^{m_3} ....p_k^{m_k}$. But by corollary 2.2.1 the number of idempotents in Z_n is 2^k. Since each $m_i \geq 1$, each $m_i + 1 \geq 2$, which yields the result.

Theorem 3.2 If D represents a D-class of Z_n, then D contains at most one idempotent.

Proof: In a D-class, either every element of D is regular or no element of D is regular.[1] Also no H-class contains more than one idempotent.[1] Since idempotents are regular, every D-class containing an idempotent is regular. Since in a regular D-class, each L-class and each R-class contains an idempotent and in Z_n, L = R = H = D = J., each H-class contains exactly one idempotent. If D is not regular, it doesn't contain an idempotent. Thus the result is obvious.

Theorem 3.3 $Z_{p_1^{m_1} p_2^{m_2} p_3^{m_3} ....p_k^{m_k}}$ is not regular if at least one $m_i > 1$.

Proof: If at least one $m_i > 1$, the number of idempotents 2^k (by corollary 2.2.1), will be strictly less than the number of distinct D-classes $\prod_i (m_i + 1)$(by theorem 3.1). But by *theorem 3.2* there will be at least one D-class which does not contain any idempotent. As the D-class which does not contain an idempotent is irregular, the semigroup $Z_{p_1^{m_1} p_2^{m_2} p_3^{m_3} ....p_k^{m_k}}$ is irregular.

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Theorem 3.4: A regular Z_n is isomorphic to some $Zp_1p_2p_3...p_k$.

Proof: By fun*dame*ntal theorem of Arithmetic(Unique Prime Factorization Theorem), every integer $n \geq 1$, can be expressed as $n = p_1^{m_1} p_2^{m_2} p_3^{m_3} ....p_k^{m_k}$, where p_1, p_2, p_3, ..., p_k are distinct primes. If Z_n is regular, each D-class of Z_n contains exactly one idempotent. By theorem 3.1 there will be $\prod_i (m_i + 1)$ distinct D-classes for Z_n. Since each D-class of regular Z_n has exactly one idempotent there will be $\prod_i (m_i + 1)$ idempotents in Z_n . Then $2^k = \prod_i (m_i + 1)$, where i = 1, 2, 3, ...k. Here the L. H. S. and R. H. S. Consists of k-factors, *therefore by comparing* them, each $(m_i + 1) = 2$. $\Rightarrow (m_i + 1) = 2 \Rightarrow m_i = 1$, for each i. Then $n = p_1p_2p_3...p_k$. Shows that there exists some $k \in Z$ such that $| Z_n | = | Zp_1p_2p_3...p_k |$. Hence they are isomorphic.

Theorem 3.5 If $n = p_1p_2p_3...p_k$, a product of distinct primes then Z_n, is an inverse semigroup.

Proof: Since Z_n, is a commutative semigroup, its idempotents commute. $Z_{p_1p_2p_3...p_k}$ is regular by corollary 2.1.1. Let e and f be two idempotents in a single L-class then e and f are two right identities in the same L-class.Thus $ef = e$ and $= f$. But in Z_n , we have $ef = fe$. Thus it means $e = f$, every L-class in regular Z_n contains exactly one idempotent. Also for Z_n, L = R = H = D = J. Thus in a D-class of regular Z_n , there will be exactly one idempotent . Let $x \in Z_n$ where $n = p_1p_2p_3...p_k$. Suppose there exists two inverses say x' and x'' for x. Then the idempotents, $xx'$and xx'' belongs to D_x and so by the above argument $' = xx''$. Similarly, the idempotents $x'x$ and $x''x$ are equal. Then

$$x' = x'xx' = x'(xx') = x'(xx'') = (x'x)x'' = (x''x)x'' = x''xx'' = x''$$

Therefore, each $x \in Z_n$ possesses a unique inverse in Z_n. Thus regular Z_n is an inverse semigroup.

IV. Conclusion

For every positive integer n we can easily identify all possible distinct D-classes of Z_n using the sets R[n] and M_x of each factor $x \neq 1$ of n. Also the Hasse Diagram of the Lattice of the partially ordered set of D-classes of Z_n can be constructed, using which we can compare the lattices of D-classes of a regular Z_n and a non-regular Z_n.

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